

A single *handwritten* A4 paper sheet with your summary of the course *is* authorized.
Computer, computing devices, phones, books, full lecture notes, friends *are not* authorized.
Double-sided sheet: check the questions behind!

Notations For $x \in \mathbb{R}^d$, the Dirac measure at x is denoted by δ_x . The p -Wasserstein distance between μ and ν is denoted by $W_p(\mu, \nu)$ for $p \in \mathbb{N}$.

1. Let $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ and $\nu = \frac{1}{3}\delta_{-1} + \frac{1}{3}\delta_2 + \frac{1}{3}\delta_3$. Let $c(x, y) = |x - y|^2$ be the cost function. Give, if they exist, the optimal (Monge) transport map between μ and ν , and optimal coupling between μ and ν .
2. Let $\mu = \mathcal{N}(0, \sigma^2)$ the centered Gaussian probability measure of variance σ^2 , and $\nu = \delta_0$ the Dirac measure at 0. Compute $W_1(\mu, \nu)$.
We recall the pdf of μ is given by $g_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{x^2}{2\sigma^2})$.
3. Let μ be the Lebesgue measure on $[0, 1]^2$ and let, for n is a fixed integer $n \in \{0, 1, 2\}$,

$$T : (x, y) \in [0, 1]^2 \rightarrow (\sqrt{x} \cos(2n\pi y), \sqrt{x} \sin(2n\pi y)) \in \mathbb{R}^2.$$

- (a) What is the image measure ν of μ by T ?
Find the values of n for which
 - (b) ν is absolutely continuous with respect to the Lebesgue measure;
 - (c) T is an optimal (Monge) transport map for the quadratic cost.
4. Let μ_0 and μ_1 two measures with density ρ_0 and ρ_1 with respect to the Lebesgue measure defined over \mathbb{R}^d . Let $c(x, y) = \|x - y\|^2$ be the quadratic cost function. Assume that μ_0 and μ_1 have finite second moments.
 - (a) What is the name of the theorem that states that the optimal transport map pushing μ_0 onto μ_1 exists? Let's denote by T this optimal transport map.
 - (b) Assume that T is bijective. Show that the Monge map pushing μ_1 onto μ_0 is T^{-1} .
 - (c) We define for any $t \in [0, 1]$, the map $T_t(x) = (1 - t)x + tT(x)$, and $\mu_t = (T_t)_\# \mu_0$. Show that $(T^{-1})_{1-t}$ maps μ_1 onto μ_t .
 - (d) Show that for any $t \in [0, 1]$, $W_2(\mu_0, \mu_t) \leq tW_2(\mu_0, \mu_1)$.
 - (e) Show that for any $t \in [0, 1]$, $W_2(\mu_1, \mu_t) \leq (1 - t)W_2(\mu_0, \mu_1)$.
 - (f) What is the value of $W_2(\mu_0, \mu_t)$?

5. Let μ be the Lebesgue measure on $[0, 2]$ and let ν be the Lebesgue measure on $[1, 3]$. Let $c(x, y) = |x - y|$ be the cost function (note that there is **no** square). Let $T_1(x) = x + 1$,

$$T_2(x) = \begin{cases} x + 2 & \text{if } x \in [0, 1], \\ x & \text{if } x \in (1, 2]. \end{cases}$$

Show that T_1 and T_2 are optimal transport maps for the optimal (Monge) transport problem between μ and ν with cost c .

6. During the practical session of an optimal transport class, a student reports the following convergence curves of the Sinkhorn algorithm in term of lack of optimality. Does the curve looks correct? If not, what is the issue? *For the sake of completeness: P is the current estimate of the optimal transport plan, a and b are the marginals, u and v are the dual variables.*

