Low Complexity Regularizations of Inverse Problems

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PhD defense

Linear Inverse Problems

Forward model

$$y = \Phi\, x_0 + w$$

Forward operator $\Phi : \mathbb{R}^n \to \mathbb{R}^q$ linear ($q \leqslant n$)

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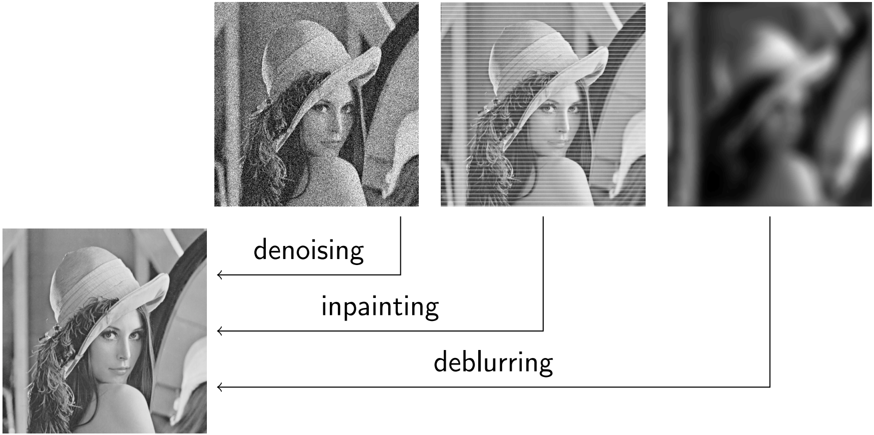

Variational Regularization

Trade-off between prior regularization and data fidelity

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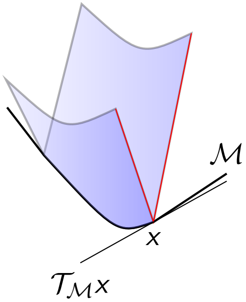
J convex, bounded from below and finite-valued function, typically non-smooth.

Partly Smooth Functions [Lewis 2002]



J is *partly smooth* at x relative to a C^2-manifold $\mathcal{M}$ if

Smoothness. J restricted to $\mathcal{M}$ is C^2 around x

Sharpness. $\forall h \in (\mathcal{T}_{\mathcal{M}}x)^{\perp}$, $t \mapsto J(x+th)$ is non-smooth at $t = 0$.

Continuity. ∂J on $\mathcal{M}$ is continuous around x.

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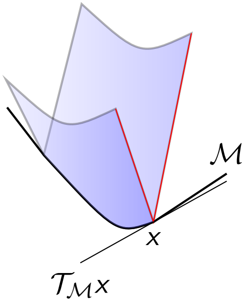

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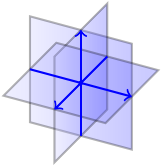
$$J, G \text{ partly smooth} \Rightarrow \begin{cases} J+G \\ J \circ D^* \text{ with } D \text{ linear operator} \\ J \circ \sigma \text{ (spectral lift)} \end{cases} \quad \text{partly smooth}$$

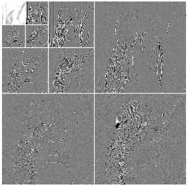
Low Complexity Models

Sparsity

$$J(x) = \sum_{i=1,\ldots,n} |x_i|$$



$$\mathcal{M}_x = \left\{ x' : \ \mathrm{supp}(x') \subseteq \mathrm{supp}(x) \right\}$$



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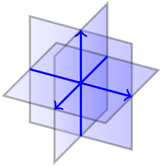

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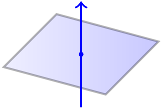

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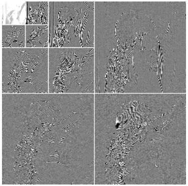

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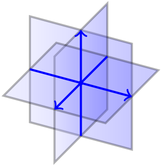

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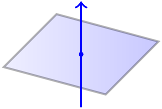
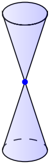


Low rank

$$J(x) = \sum_{i=1,\ldots,n} |\sigma_i(x)|$$



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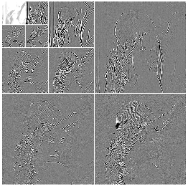
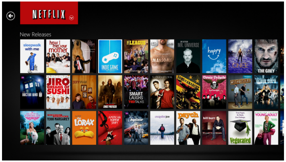


$$\mathcal{M}_x = \{x' \,:\, \mathrm{rank}(x') = \mathrm{rank}(x)\}$$



Contributions of the Thesis

$$x^\star \in \underset{x \in \mathbb{R}^n}{\operatorname{Argmin}} \frac{1}{2}\|y - \Phi x\|^2 + \lambda J(x) \qquad (P_{y,\lambda})$$

I. Noise ℓ^2 Stability

Ensure that

$$\|x^\star - x_0\|_2 = O(\|w\|_2)$$

II. Stable Manifold Identification

Ensure that

$$x^\star \in \mathcal{M}_{x_0}$$

III. Local Differentiability

Variation of $x^\star$ as a function of the observations y.

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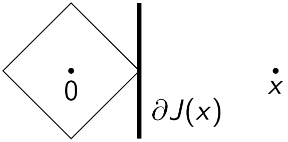
$$\partial J(x) = \left\{ \eta \in \mathbb{R}^n : \; \forall x', J(x) - J(x') \geqslant \langle \eta, \, x - x' \rangle \right\}$$



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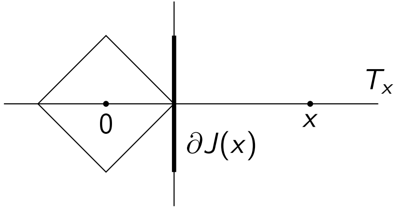

Model tangent space $T_x = \mathrm{VectHull}(\partial J(x))^\perp$

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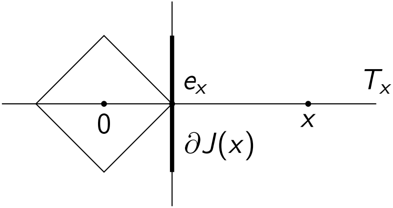

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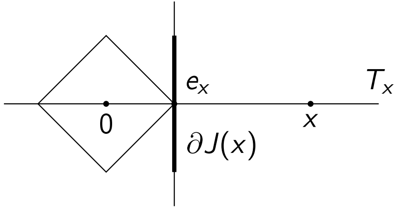

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Sparsity $\|\cdot\|_1$

$$T_x = \{\eta : \ \mathrm{supp}(\eta) \subseteq \mathrm{supp}(x)\}$$
$$e_x = \mathrm{sign}(x)$$

Trace Norm $\|\cdot\|_*$

$$T_x = \{\eta : \ U_\perp^* \eta V_\perp = 0\}$$
$$e_x = UV^*$$

Dual Certificates

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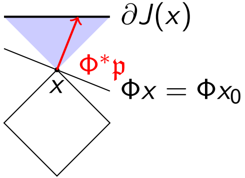

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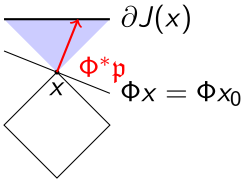

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Non-degenerate source condition

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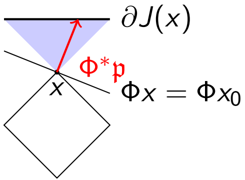

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Theorem

Assume

$$\Phi^* \mathfrak{p} \in \operatorname{ri} \partial J(x_0) \quad \text{and} \quad \operatorname{Ker} \Phi \cap T_{x_0} = \{0\}.$$

Choosing $\lambda = c\|w\|_2$, $c > 0$, for any minimizer $x^\star$ of $(P_{y,\lambda})$

$$\|x^\star - x_0\|_2 \leqslant C(c, \mathfrak{p})\|w\|_2 \ .$$

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[Grasmair et al. 2010]: ℓ^1

[Grasmair 2011]: $J(x^\star - x_0) = O(\|w\|_2)$

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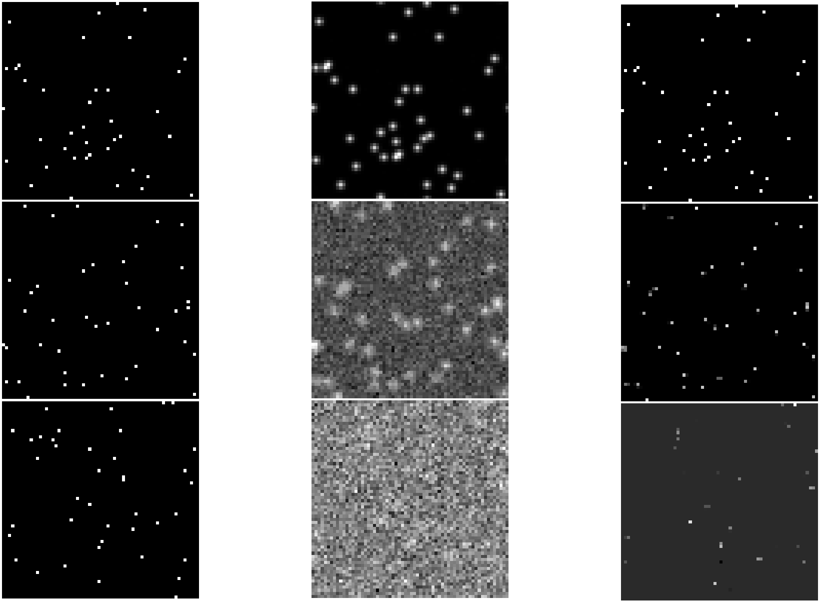

x_0 y $x^\star$

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Proposition

Assume $\operatorname{Ker} \Phi \cap T = \{0\}$ where $T = T_{x_0}$. Then,

$$\mathfrak{p}_F = \Phi_T^{+,*} e$$

where $\Phi_T = \Phi P_T$. Moreover,

$$\mathfrak{p}_F \in \operatorname{ri} \partial J(x_0) \Rightarrow \mathfrak{p}_F = \mathfrak{p}_0$$

Manifold Selection

Theorem

Assume J is partly smooth at x_0 relative to $\mathcal{M}$. If

$$\Phi^* \mathfrak{p}_F \in \operatorname{ri} \partial J(x_0) \quad \text{and} \quad \operatorname{Ker} \Phi \cap T_{x_0} = \{0\}.$$

There exists $C > 0$ such that if

$$\max(\lambda, \|w\|/\lambda) \leqslant C,$$

the unique solution $x^\star$ of $(P_{y,\lambda})$ satisfies

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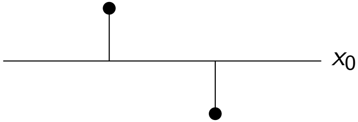
[Fuchs 2004]: ℓ^1

[Bach 2008]: $\ell^1 - \ell^2$ and nuclear norm.

Sparse Spike Deconvolution



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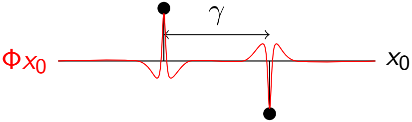

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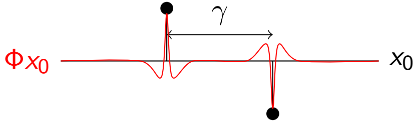
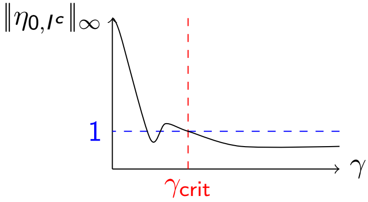


$$\Phi^* \eta_F \in \operatorname{ri} \partial J(x_0) \Leftrightarrow \|\Phi_{I^c}^{+,*} \Phi_I \operatorname{sign}(x_{0,I})\|_\infty < 1 \Leftrightarrow \text{stable recovery}$$

$$I = \operatorname{supp}(x_0)$$



1D Total Variation and Jump Set

$$J = \|\nabla_d \cdot \|_1, \mathcal{M}_x = \left\{x' : \ \mathsf{supp}(\nabla_d x') \subseteq \mathsf{supp}(\nabla_d x)\right\}, \Phi = \mathrm{Id}$$

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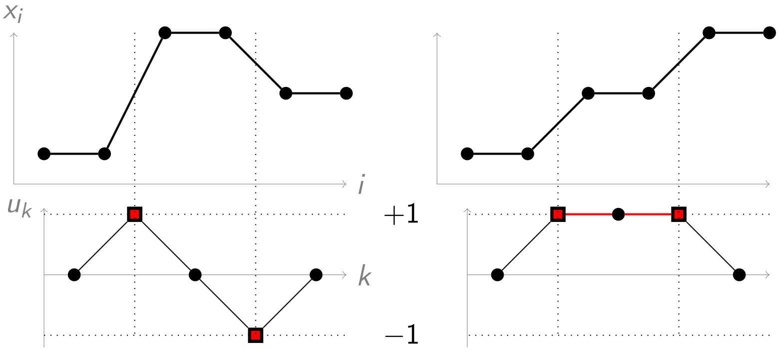

stable jump

unstable jump

$$\Phi^* \mathfrak{p}_F = \operatorname{div} u$$

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Sensitivity Analysis of Optimization Problems

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If J is smooth, first-order conditions

$$\Phi^*(\Phi x^\star - y) + \lambda \nabla J(x^\star) = 0$$

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Implicit function theorem $\to$ If

$$\Gamma = \Phi^*\Phi + \lambda \mathrm{D}^2 J(x^\star)$$

is invertible, then on a neighborhood, there exists $\bar{y} \mapsto \tilde{x}(\bar{y})$ (C^1) such that $\tilde{x}(\bar{y})$ solution of $(P_{\bar{y},\lambda})$, $\tilde{x}(y) = x^\star$ and

$$\mathrm{D}\tilde{x}(y) = \Gamma^{-1}\Phi^*$$

Objective

$$x^\star \in \underset{x \in \mathbb{R}^n}{\mathrm{Argmin}} \ \frac{1}{2}\|y - \Phi x\|^2 + \lambda J(x) \qquad (P_{y,\lambda})$$

If J is smooth, first-order conditions

$$\Phi^*(\Phi x^\star - y) + \lambda \nabla J(x^\star) = 0$$

Implicit function theorem $\rightarrow$ If

$$\Gamma = \Phi^*\Phi + \lambda \mathrm{D}^2 J(x^\star)$$

is invertible, then on a neighborhood, there exists $\bar{y} \mapsto \tilde{x}(\bar{y})$ (C^1) such that $\tilde{x}(\bar{y})$ solution of $(P_{\bar{y},\lambda})$, $\tilde{x}(y) = x^\star$ and

$$\mathrm{D}\tilde{x}(y) = \Gamma^{-1}\Phi^*$$

Issues

1. What if J not differentiable ?
2. What if Γ is not invertible ?

Transition Space

J partly smooth with linear manifolds and $(T_x)_{x \in \mathbb{R}^n}$ finite

$$\mathcal{H} = \bigcup_{T \in (T_x)_{x \in \mathbb{R}^n}} \mathrm{bd}\left(\Pi_1 \left\{ (y, x_T) : \frac{1}{\lambda} \Phi_T^* (\Phi_T x_T - y) \in \mathrm{rbd}\, \partial J(x_T) \right\} \right)$$

Π_1: projection on the first component

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Lemma

If J is semi-algebraic, $\mathcal{H}$ is semi-algebraic and has zero measure w.r.t Lebesgue measure on $\mathbb{R}^q$.

Semi-algebraic set: boolean combination of polynomial equalities and inequalities
Semi-algebraic function: function with a semi-algebraic graph

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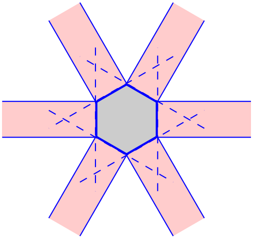

$J = \|\cdot\|_1, \Phi \in \mathbb{R}^{2\times 3}$

Semi-algebraic set: boolean combination of polynomial equalities and inequalities

Semi-algebraic function: function with a semi-algebraic graph

ex: polyhedral functions $\Rightarrow$ $\mathcal{H}$ is a union of subspaces

Local Behavior

$$x^\star \in \operatorname*{Argmin}_{x \in \mathbb{R}^n} \frac{1}{2}\|y - \Phi x\|^2 + \lambda J(x) \qquad (P_{y,\lambda})$$

Theorem

Let $T = T_{x^\star}$. Assume

$$y \notin \mathcal{H} \quad \text{and} \quad \operatorname{Ker}\Phi \cap \operatorname{Ker} \mathrm{D}^2 J_T(x^\star) = \{0\}.$$

Then, there exists, on a neighborhood $\mathcal{O}$ of y, a mapping $\tilde{x}$ s.t.

1. For every $\bar{y} \in \mathcal{O}$, $\tilde{x}(\bar{y})$ is a solution of $(\mathcal{P}_{\bar{y},\lambda})$ and $\tilde{x}(y) = x^\star$.
2. The mapping $\tilde{x}$ is $\mathrm{C}^1(\mathcal{O})$ and

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Contributions of the Thesis

$$x^\star \in \underset{x \in \mathbb{R}^n}{\mathrm{Argmin}} \; \frac{1}{2}\|y - \Phi x\|^2 + \lambda J(x) \qquad (P_{y,\lambda})$$

I. Noise ℓ^2 Stability

Ensure that

$$\|x^\star - x_0\|_2 = O(\|w\|_2)$$

II. Stable Manifold Identification

Ensure that

$$x^\star \in \mathcal{M}_{x_0}$$

III. Local Differentiability

Variation of $x^\star$ as a function of the observations y.

IV. Unbiased Risk Estimation

Select the optimal parameter λ.

Variations on a theme:
Sensitivity Analysis of Optimization Problems

Objective

$$x^\star(y) \in \underset{x \in \mathbb{R}^n}{\text{Argmin}} \; \frac{1}{2}\|y - \Phi x\|^2 + \lambda J(x) \qquad (P_{y,\lambda})$$

Stochastic linear model

$$Y = \Phi x_0 + W \sim \mathcal{N}(\Phi x_0, \sigma^2 \mathrm{Id})$$

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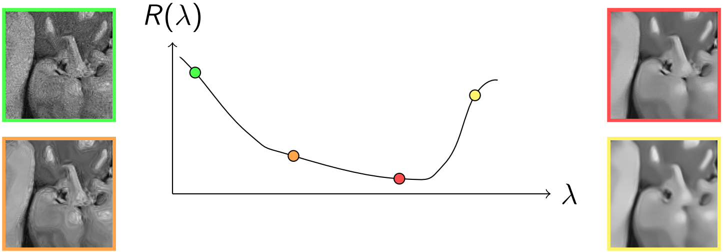
(Estimation) risk

$$R(\lambda) = \mathbb{E}_W \left[\|x^\star(Y) - x_0\|_2^2 \right]$$



Stein's Unbiased Risk Estimation

Prediction : $\mu(y) = \Phi x^\star(y)$ (single-valued even if $x^\star$ not)

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If μ is Lipschitz, then

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$\rightarrow$ must have $\mathrm{div}(\mu)$ in closed form Lebesgue-almost everywhere

Practical Computation

Theorem

If J is semi-algebraic, then

$$df(Y) = \operatorname{tr}\left[\Phi_T(\Phi_T^*\Phi_T + \lambda \mathrm{D}^2 J_T(x^\star(Y)))^{-1}\Phi_T\right] \quad \text{a.e.}$$

where $T = T_{x^\star}$ and $x^\star$ a solution s.t. $\operatorname{Ker}\Phi \cap \operatorname{Ker}\mathrm{D}^2 J_T(x^\star) = \{0\}$.

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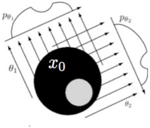
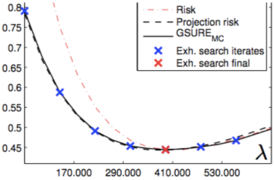
Projected GSURE: unbiased estimate of $\|P^\perp_{\operatorname{Ker}\Phi}(x^\star(Y) - x_0)\|^2$

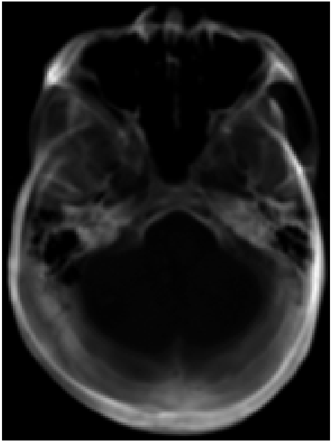
An Application to Tomography

Φ sub-sampled Radon transform (16)

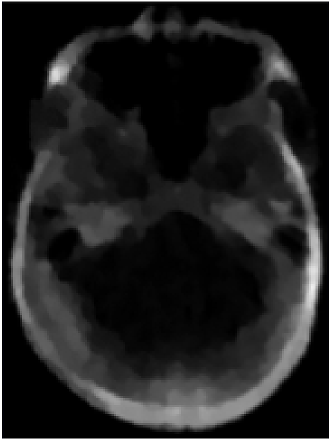
$J = \|\nabla_d \cdot\|_{\mathcal{B}}$







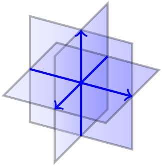
x_0



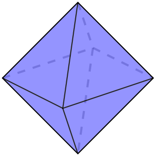
$x^\star_{\lambda^\star}$

Summary

Partial smoothness: encodes models using singularities



$\longrightarrow$



Performance

- $\odot$ ℓ^2 error
- $\odot$ model stability

Parameter selection

- $\odot$ local behavior
- $\odot$ unbiased risk estimation

$\Longrightarrow$ **Sensitivity analysis**

Future Work

Extended-valued functions: minimization under constraints

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|y - \Phi x\|^2 + \lambda J(x) \quad \text{subject to} \quad x \geqslant 0$$

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Compressed sensing: Optimal bounds for partly smooth regularizers