Characterizing the maximum parameter of the total-variation denoising through the pseudo-inverse of the divergence

Charles-Alban Deledalle Nicolas Papadakis
IMB, CNRS, Bordeaux INP
Université de Bordeaux, Talence, France
Email: firstname.lastname@math.u-bordeaux.fr

Joseph Salmon
LTCI, Télécom ParisTech
Université Paris-Saclay, France
Email: joseph.salmon@telecom-paristech.fr

Samuel Vaiter
IMB, CNRS
Université de Bourgogne, Dijon, France
Email: samuel.vaiter@u-bourgogne.fr

***Abstract*—We focus on the maximum regularization parameter for anisotropic total-variation denoising. It corresponds to the minimum value of the regularization parameter above which the solution remains constant. While this value is well know for the Lasso, such a critical value has not been investigated in details for the total-variation. Though, it is of importance when tuning the regularization parameter as it allows fixing an upper-bound on the grid for which the optimal parameter is sought. We establish a closed form expression for the one-dimensional case, as well as an upper-bound for the two-dimensional case, that appears reasonably tight in practice. This problem is directly linked to the computation of the pseudo-inverse of the divergence, which can be quickly obtained by performing convolutions in the Fourier domain.**

I. Introduction

We consider the reconstruction of a d-dimensional signal (in this study $d = 1$ or 2) from its noisy observation $y = x + w \in \mathbb{R}^n$ with $w \in \mathbb{R}^n$. Anisotropic TV regularization writes, for $\lambda > 0$, as [1]

$$x^\star = \operatorname*{argmin}_{x \in \mathbb{R}^n} \frac{1}{2}\|y - x\|_2^2 + \lambda \|\nabla x\|_1 \tag{1}$$

with $\nabla x \in \mathbb{R}^{dn}$ being the concatenation of the d components of the discrete periodical gradient vector field of x, and $\|\nabla x\|_1 = \sum_i |(\nabla x)_i|$ being a sparsity promoting term. The operator ∇ acts as a convolution which writes in the one dimensional case ($d = 1$)

$$\nabla = F^+ \operatorname{diag}(K_\downarrow) F \quad \text{and} \quad \operatorname{div} = F^+ \operatorname{diag}(K_\uparrow) F \tag{2}$$

where $\operatorname{div} = -\nabla^\top$ (where $^\top$ denotes the adjoint), $F : \mathbb{R}^n \mapsto \mathbb{C}^n$ is the Fourier transform, $F^+ = \operatorname{Re}[F^{-1}]$ is its pseudo-inverse and $K_\downarrow \in \mathbb{C}^n$ and $K_\uparrow \in \mathbb{C}^n$ are the Fourier transforms of the kernel functions performing forward and backward finite differences respectively. Similarly, we define in the two dimensional case ($d = 2$)

$$\nabla = \begin{pmatrix} F^+ & 0 \\ 0 & F^+ \end{pmatrix} \begin{pmatrix} \operatorname{diag}(K_\downarrow) \\ \operatorname{diag}(K_\rightarrow) \end{pmatrix} F \tag{3}$$

$$\text{and} \quad \operatorname{div} = F^+ \begin{pmatrix} \operatorname{diag}(K_\uparrow) & \operatorname{diag}(K_\leftarrow) \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} \tag{4}$$

where $K_\rightarrow \in \mathbb{C}^n$ and $K_\leftarrow \in \mathbb{C}^n$ (resp. $K_\downarrow \in \mathbb{C}^n$ and $K_\uparrow \in \mathbb{C}^n$) perform forward and backward finite difference in the horizontal (resp. vertical) direction.

II. General case

For the general case, the following proposition provides an expression of the maximum regularization parameter $\lambda_{\max}$ as the solution of a convex but non-trivial optimization problem (direct consequence of the Karush-Khun-Tucker condition).

Proposition 1. *Define for $y \in \mathbb{R}^n$,*

$$\lambda_{\max} = \min_{\zeta \in \operatorname{Ker}[\operatorname{div}]} \| \operatorname{div}^+ y + \zeta \|_\infty \tag{5}$$

where div^+ *is the Moore-Penrose pseudo-inverse of* div *and* $\operatorname{Ker}[\operatorname{div}]$ *its null space. Then, $x^\star = \frac{1}{n} \mathbb{1}_n \mathbb{1}_n^\top y$ if and only if $\lambda \geqslant \lambda_{\max}$.*

III. One dimensional case

In the 1d case, $\operatorname{Ker}[\operatorname{div}] = \operatorname{Span}(\mathbb{1}_n)$ and thus the optimization problem can be solved by computing div^+ in the Fourier domain, in $O(n \log n)$ operations, as shown in the next corollary.

Corollary 1. *For $d = 1$, $\lambda_{\max} = \frac{1}{2}[\max(\operatorname{div}^+ y) - \min(\operatorname{div}^+ y)]$,*

$$\textit{where} \quad \operatorname{div}^+ = F^+ \operatorname{diag}(K_\uparrow^+) F \tag{6}$$

$$\textit{and} \quad (K_\uparrow^+)_i = \begin{cases} \frac{(K_\uparrow)_i^*}{|(K_\uparrow)_i|^2} & \textit{if} \quad |(K_\uparrow)_i|^2 > 0 \\ 0 & \textit{otherwise} \end{cases},$$

and * denotes the complex conjugate.

Note that the condition $|(K_\uparrow)_i|^2 > 0$ is satisfied everywhere except for the zero frequency. In the non-periodical case, div is the incidence matrix of a tree whose pseudo-inverse can be obtained following [2].

IV. Two dimensional case

In the 2d case, $\operatorname{Ker}[\operatorname{div}]$ is the orthogonal of the vector space of vector fields satisfying Kirchhoff's voltage law on all cycles of the periodical grid. Its dimension is $n+1$. It follows that the optimization problem becomes much harder. Since our motivation is only to provide an approximation of $\lambda_{\max}$, we propose to compute an upper-bound in $O(n \log n)$ operations thanks to the following corollary.

Corollary 2. *For $d = 2$, $\lambda_{\max} \leqslant \underbrace{\frac{1}{2}[\max(\operatorname{div}^+ y) - \min(\operatorname{div}^+ y)]}_{\lambda_{\text{bnd}}}$,*

$$\textit{where} \quad \operatorname{div}^+ = \begin{pmatrix} F^+ & 0 \\ 0 & F^+ \end{pmatrix} \begin{pmatrix} \operatorname{diag}(\tilde{K}_\uparrow^+) \\ \operatorname{diag}(\tilde{K}_\leftarrow^+) \end{pmatrix} F, \quad \textit{and} \tag{7}$$

$$(\tilde{K}_\uparrow^+)_i = \begin{cases} \frac{(K_\uparrow)_i^*}{|(K_\uparrow)_i|^2 + |(K_\leftarrow)_i|^2} & \textit{if} \quad |(K_\uparrow)_i|^2 + |(K_\leftarrow)_i|^2 > 0 \\ 0 & \textit{otherwise} \end{cases},$$

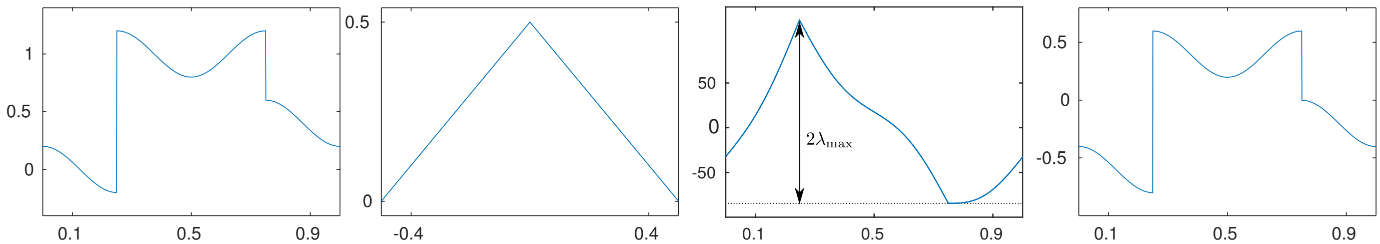
$$(\tilde{K}_\leftarrow^+)_i = \begin{cases} \frac{(K_\leftarrow)_i^*}{|(K_\uparrow)_i|^2 + |(K_\leftarrow)_i|^2} & \textit{if} \quad |(K_\uparrow)_i|^2 + |(K_\leftarrow)_i|^2 > 0 \\ 0 & \textit{otherwise} \end{cases}.$$

Note that the condition $|(K_\uparrow)_i|^2 + |(K_\leftarrow)_i|^2 > 0$ is again satisfied everywhere except for the zero frequency. Remark also that this result can be straightforwardly extended to the case where $d > 2$.

V. Results and discussion

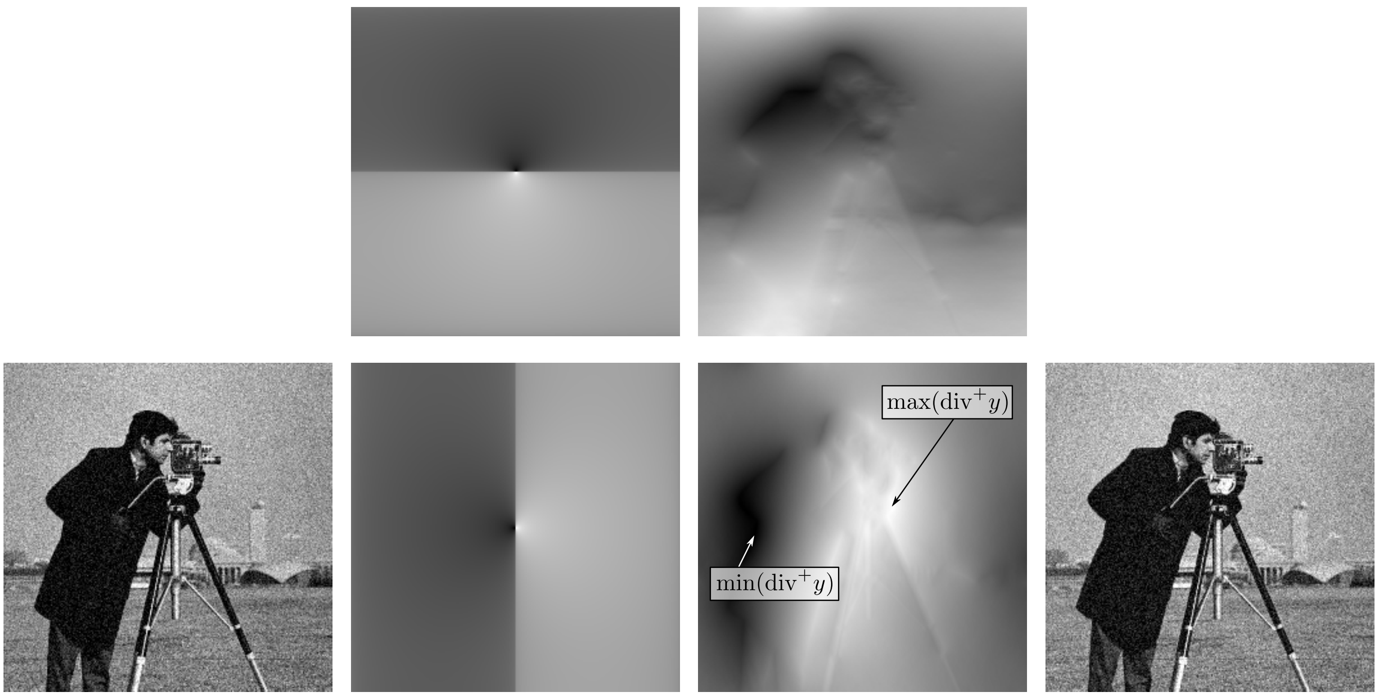
Figure 1 and 2 provide illustrations of the computation of $\lambda_{\max}$ and λ_{bnd} on a 1d signal and a 2d image respectively. The convolution kernel is a simple triangle wave in the 1d case but is more complex in the 2d case. The operator $\operatorname{div} \operatorname{div}^+$ is in fact the projector onto the space of zero-mean signals, i.e., Im[div]. Figure 3 illustrates the evolution of $x^\star$ with respect to λ (computed with the algorithm of [3]). Our upper-bound λ_{bnd} (computed in ∼5ms) appears to be reasonably tight ($\lambda_{\max}$ computed in ∼25s with [3] on Problem (5)).

Future work will concern the generalization of these results to other ℓ_1 sparse analysis regularization and to ill-posed inverse problems.



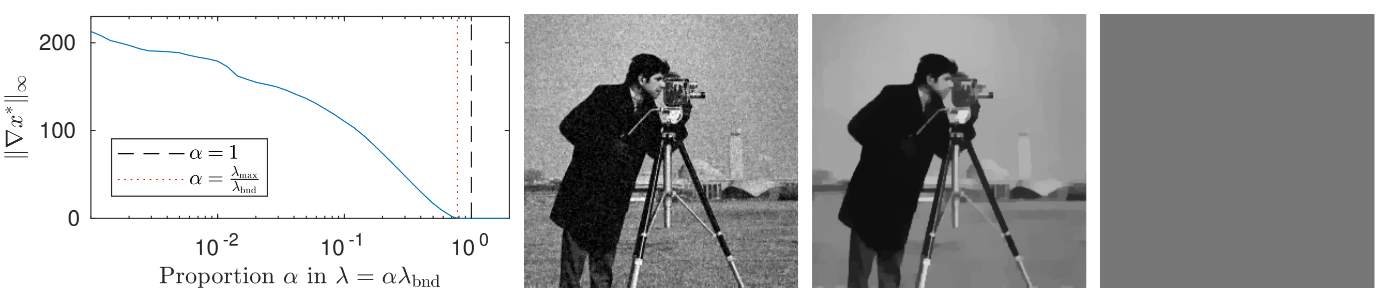
(a) y (b) $F^+K_\uparrow^+$ (c) $\operatorname{div}^+ y$ (d) $\operatorname{div}\operatorname{div}^+ y$

Fig. 1. (a) A 1d signal y. (b) The convolution kernel $F^+K_\uparrow^+$ that realizes the pseudo inversion of the divergence. (c) The signal $\operatorname{div}^+ y$ on which we can read the value of $\lambda_{\max}$. (d) The signal $\operatorname{div}\operatorname{div}^+ y$ showing that one can reconstruct y from $\operatorname{div}^+ y$ up to its mean component.



(a) y (range $[0, 255]$) (b) $F^+\tilde{K}_\uparrow^+$, $F^+\tilde{K}_\leftarrow^+$ (in power scale) (c) $\operatorname{div}^+ y$ (d) $\operatorname{div}\operatorname{div}^+ y$ (range $[-119, 136]$)

Fig. 2. (a) A 2d signal y. (b) The convolution kernels $F^+K_\uparrow^+$ and $F^+\tilde{K}_\leftarrow^+$ that realizes the pseudo inversion of the divergence. (c) The two coordinates of the vector field $\operatorname{div}^+ y$ on which we can read the upper-bound λ_{bnd} of $\lambda_{\max}$. (d) The image $\operatorname{div}\operatorname{div}^+ y$ showing again that one can reconstruct y from $\operatorname{div}^+ y$ up to its mean component.



(a) (b) $\lambda = 10^{-3}\lambda_{\text{bnd}}$ (c) $\lambda = 10^{-2}\lambda_{\text{bnd}}$ (d) $\lambda = \lambda_{\text{bnd}}$

Fig. 3. (a) Evolution of $\|\nabla x^\star\|_\infty$ as a function of λ. (b), (c), (d) Results $x^\star$ of the periodical anisotropic total-variation for three different values of λ.

References

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