Strongly convex function

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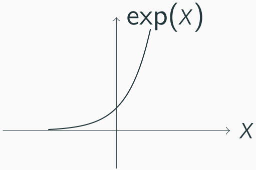
Strongly convex function

Convex function

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

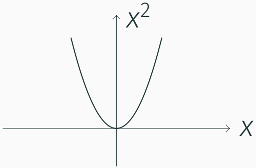


Definition

f is a strongly convex function of modulus $\mu > 0$ if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) - \frac{1}{2}\mu\lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in \mathbf{dom}\, f$, $\lambda \in [0,1]$.



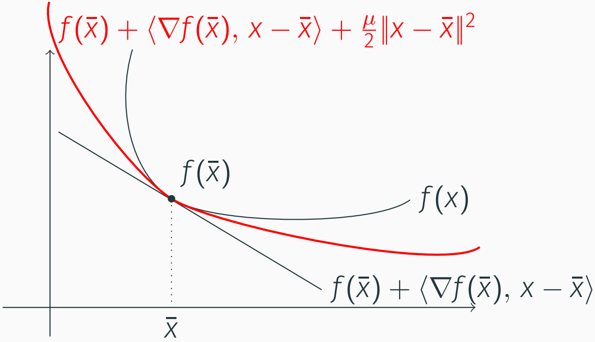
f μ-strongly convex $\Leftrightarrow f - \frac{\mu}{2}\|\cdot\|_2^2$ convex.
f μ-strongly convex $\Rightarrow f$ has a unique minimizer.

Strong convexity and differentiability

Proposition

A differentiable function f is μ-strongly convex, with $\mu > 0$, iff

$$\forall x, \bar{x} \in \operatorname{dom} f, \quad f(x) \geq f(\bar{x}) + \langle \nabla f(\bar{x}), x - \bar{x} \rangle + \frac{\mu}{2}\|x - \bar{x}\|^2$$



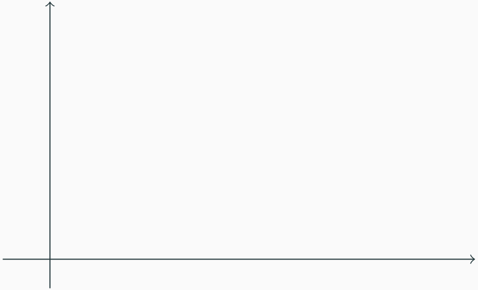
Strong convexity $\implies$ lower bounds

Proposition

If f is μ-strongly convex and differentiable, then the following lower bound holds

$$\forall x \in \mathbb{R}^d, \quad f(x) \geq f(x^\star) + \frac{\mu}{2}\|x - x^\star\|^2$$

where $x^\star = \operatorname{argmin} f$.



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