Smooth function *a.k.a.* function with Lipschitz gradient

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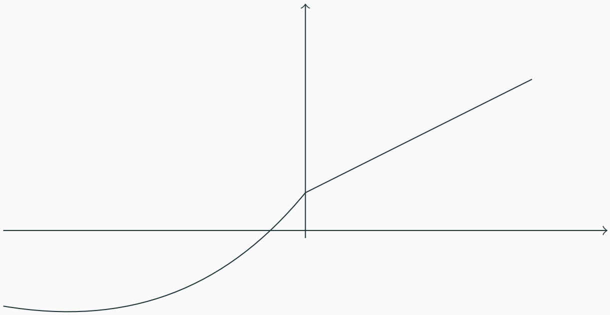
Lipschitz continuity

Definition

$\phi : \Omega \subseteq E \to F$ is L-Lipschitz continuous if

$$\forall x, y \in \Omega, \quad \|\phi(x) - \phi(y)\|_E \leq L\|x - y\|_F.$$



(live)

Smooth function: quadratic upper-bound

⚠ "smooth" refers to a lot of different concept in mathematics.

Definition

A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be L-smooth if ∇f is L-Lipschitz

$$\forall x, y \in \operatorname{dom} f, \quad \|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|.$$

Smooth function: quadratic upper-bound

L-smooth: $\forall x, y \in \mathbf{dom}\, f, \|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$.

Proposition

Let f a L-smooth function. Then, for all $x, y \in \mathrm{dom} f$,

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \leq L\|x - y\|^2. \quad (1)$$

Moreover, if $\mathrm{dom} f$ is convex, then (1) is equivalent to

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|x - y\|^2, \quad \forall x, y \in \mathrm{dom} f. \quad (2)$$