What is optimization ?

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Optimization?

Optimization problem

$$\inf f(x) \quad \text{subject to} \quad x \in \Omega,$$

- Domain $\Omega \subseteq \mathcal{X}$.
- Optimization variable x.
- Cost (or objective, energy, loss) function $f : \Omega \to \mathbb{R}$.

Existence? Uniqueness? Algorithm?

Optimization?

Optimization problem

$$\min f(x) \quad \text{subject to} \quad x \in \Omega,$$

- Domain $\Omega \subseteq \mathcal{X}$.
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- Cost (or objective, energy, loss) function $f : \Omega \to \mathbb{R}$.
- Optimal solution(s)

$$x^\star \in \operatorname{argmin} f(x) \quad \text{subject to} \quad x \in \Omega,$$

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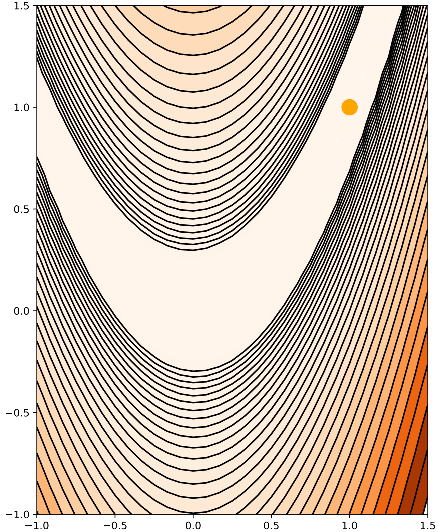
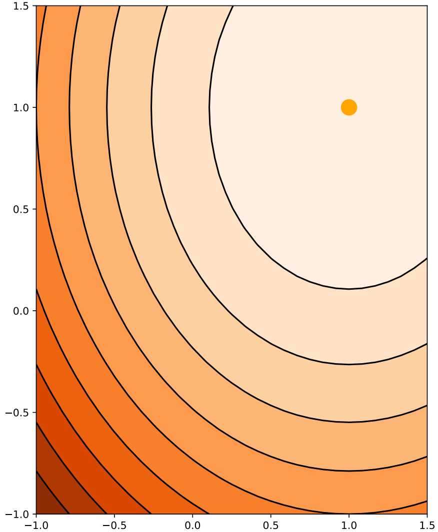
Existence? Uniqueness? Algorithm?

Example: Toy examples





Example: Risk minimization

- Joint distribution $\mathbb{P}$ on input $x \in \mathcal{X}$, output $y \in \mathcal{Y}$.
- Loss function $\ell : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ (*e.g.*, ℓ^2-norm $\ell(y, y') = (y - y)^2$).

Statistical (or expected) risk

$$\inf_{\phi:\mathcal{X}\to\mathcal{Y}} \mathcal{R}(\phi) = \mathbb{E}_{(x,y)\sim\mathbb{P}}[\ell(y, \phi(x))].$$

- Domain: (Measurable) functions from inputs $\mathcal{X}$ to outputs $\mathcal{Y}$.
- Variable: prediction function $\phi : \mathcal{X} \to \mathcal{Y}$.
- Cost: expected statistical risk $\mathcal{R}$.

Check your statistical learning 101 class!

Example: Empirical Risk minimization

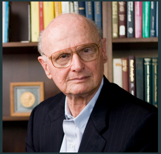
- n examples in a training set $(x_i, y_i) \in \mathcal{X} \times \mathcal{Y}$.
- Loss function $\ell : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ (*e.g.*, ℓ^2-norm $\ell(y, y') = (y - y)^2$).

Empirical risk minimization (ERM)

$$\inf_{\phi:\mathcal{X}\to\mathcal{Y}} \hat{\mathcal{R}}(\phi) = \frac{1}{n}\sum_{i=1}^{n} \ell(y_i, \phi(x_i)).$$

- Domain: (Measurable) functions from inputs $\mathcal{X}$ to outputs $\mathcal{Y}$.
- Variable: prediction function $\phi : \mathcal{X} \to \mathcal{Y}$.
- Cost: expected empirical risk $\hat{\mathcal{R}}$.

Example: Portfolio optimization



H. Markowitz (1927–)

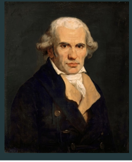
- Assets with returns $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$.
- Known expected returns $\mu = (\mu_1, \ldots, \mu_n)$.
- Risk in/between (covariance matrix positive definite) $\Sigma \in \mathbb{R}^{n \times n}$.

Portfolio with minimal variance

$$\inf_{x \in \mathbb{R}^n} f(x) \stackrel{\text{def.}}{=} \langle \Sigma x, x \rangle \quad \text{subject to} \quad \begin{cases} \mathbb{E}(\langle x, a \rangle) = \langle x, \mu \rangle = r \\ \langle x, 1 \rangle = 1. \end{cases}$$

- Domain: Portfolio with expected return μ
- Variable: Portfolio $x \in \mathbb{R}^n$
- Cost: Variance of the portfolio $\langle \Sigma x, x \rangle$

Example: Optimal transport



G. Monge (1746–1818)

- Probability measures μ on X and ν on Y.
- Cost function $c : X \times Y \to \overline{\mathbb{R}}$.

Optimal transport (Monge formulation)

$$\inf_{T:X\to Y} f(T) = \int_X c(x, T(x))d\mu(x) \quad \text{subject to} \quad T_\sharp \mu = \nu.$$

- Domain: Set of admissible transport map $T_\sharp \mu = \nu$.
- Variable: Transport map T.
- Cost: Cost of transport from μ to ν.