Finite differences

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Finite differences

For $f : \mathbb{R} \to \mathbb{R}$ and $\varepsilon > 0$ small enough,

$$f'(x) \approx \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \overset{\text{def.}}{=} \Delta_\varepsilon f(x).$$

Proposition

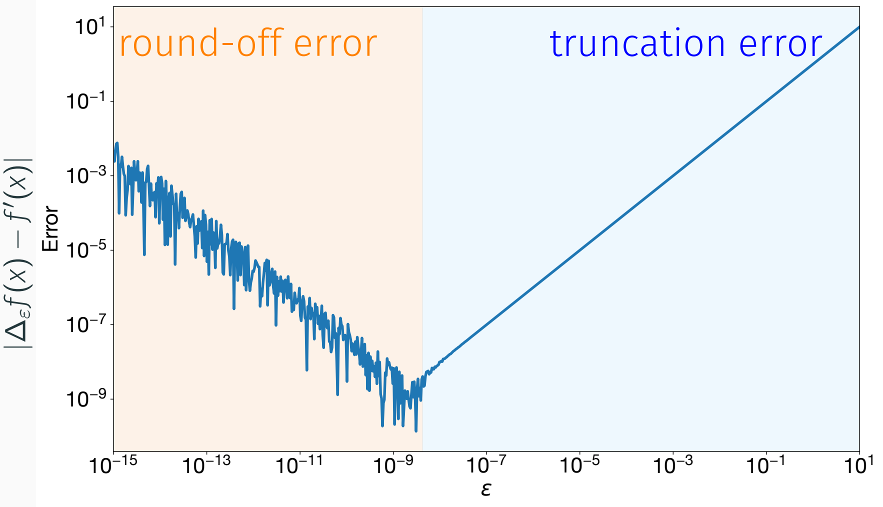
Let $f : \mathbb{R} \to \mathbb{R}$ a C^2 function around $\bar{x}$. Then, there exists $C > 0$ such that for $\varepsilon > 0$ small enough, one has

$$\left| \frac{f(\bar{x}+\varepsilon) - f(\bar{x})}{\varepsilon} - f'(\bar{x}) \right| \leq C\varepsilon \quad \textit{(}O(\varepsilon)\textit{ approximation error)}.$$

Simple to implement

```
def diff(f, x, eps=1e-5):
    return (f(x + eps) - f(x)) / eps
```

Shortcoming 1: round-off and truncation



Shortcoming 2: linear complexity w.r.t dimensions

$f : \mathbb{R}^n \to \mathbb{R}$

$$\nabla f(x) \approx \left.\begin{pmatrix} \dfrac{\overbrace{f(x + \varepsilon \mathbf{e}_1) - f(x)}^{\text{2 calls}}}{\varepsilon} \\ \vdots \\ \dfrac{f(x + \varepsilon \mathbf{e}_n) - f(x)}{\varepsilon} \end{pmatrix}\right\} n \text{ elements}$$

Linear cost:

$$\mathrm{Cost}(\nabla f) = 2n \cdot \mathrm{Cost}(f)$$