

# Minimizer and its existence

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# Local and global minimizers

$$\min_{x \in \Omega} f(x) \quad (1)$$

We say that  $x^* \in \Omega$  is

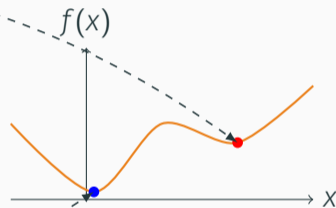
- a **local** minimizer of (1), if there exists a neighborhood  $\mathcal{O}$  of  $x^*$  such that

$$\forall x \in \Omega \cap \mathcal{O}, \quad f(x) \geq f(x^*)$$

- a **(global)** minimizer if

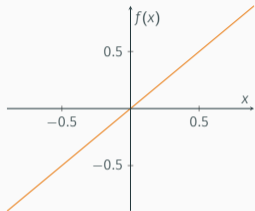
$$\forall x \in \Omega, \quad f(x) \geq f(x^*)$$

The set of global minimizers of  $f$  is denoted  $\operatorname{argmin} f$



## Existence of a minimizer: univariate case

$$f(x) = x$$

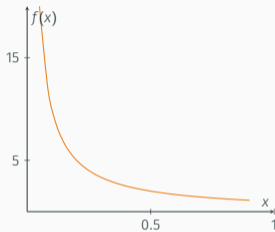


Unbounded from below

$$\inf f = -\infty$$

$$\operatorname{argmin} f = \emptyset$$

$$f(x) = 1/x \quad (x > 0)$$

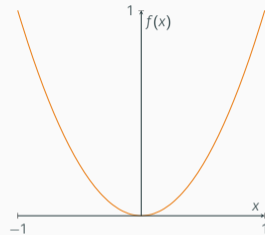


Bounded but not achieved

$$\inf f = 0$$

$$\operatorname{argmin} f = \emptyset$$

$$f(x) = x^2$$



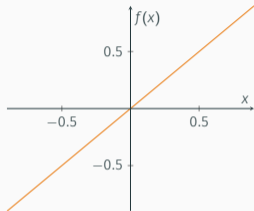
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## Existence of a minimizer: univariate case

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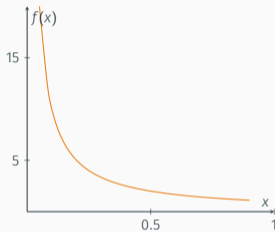


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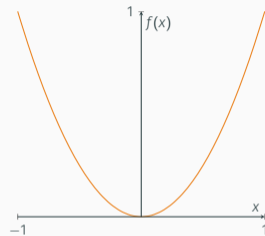


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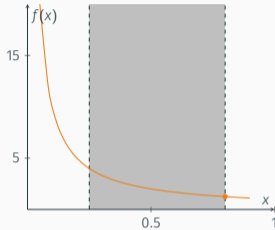


Bounded and achieved

$$\inf f = 0$$

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# Favorable case: compactness *and* continuity



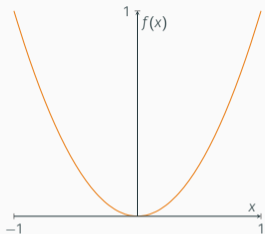
## Theorem

If  $f$  is *continuous* and  $\Omega$  is a *non-empty compact*, then there exists at least one global minima of  $f$  on  $\Omega$ .

## Proof idea

Use BOLZANO–WEIERSTRASS theorem

## Favorable case: coercivity *and* continuity



Coercive function:  $\lim_{\|x\| \rightarrow +\infty} f(x) = +\infty$ .

- $x \mapsto x$  is coercive
- $x \mapsto 1/x$  is not coercive
- $x \mapsto e^x$  is ...

### Theorem

If  $f$  is *continuous* and *coercive*, then there exists at least one global minima of  $f$  on  $\mathbb{R}^d$ .

### Proof idea

Apply the compact+continuous thm to a closed ball  $\bar{B}(0, M)$  thanks to coercivity.