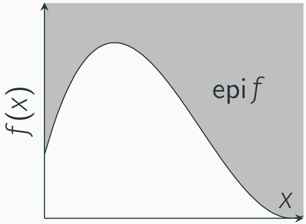
Convex function

Samuel Vaiter

Created: 2023-11-08.
Last update: 2023-11-08.
Status: inprogress.

Graph & Epigraph



Definition (Graph)

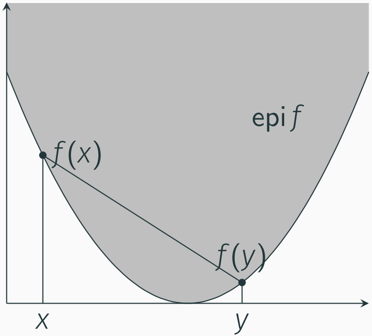
$$\mathrm{graph}\, f = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \;:\; t = f(x)\}$$

Definition (Epigraph)

$$\mathrm{epi}\, f = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \;:\; t \geq f(x)\}$$

Convex function

Convex: the line joining $f(x)$ and $f(y)$ lies above the graph of f between x and y



Definition (Convex function)

A function $f : \mathbb{R}^p \to \overline{\mathbb{R}}$ such that $f \neq +\infty$ is *convex* iff its epigraph $\mathbf{epi}\, f$ is convex.

Jensen's inequality

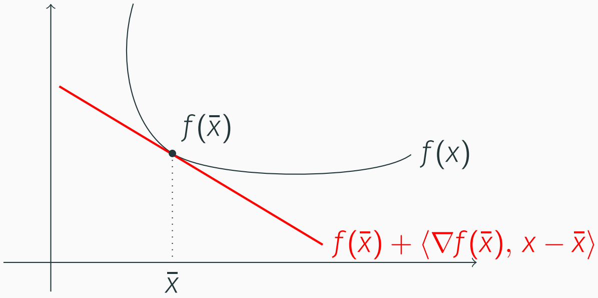
$$\forall x, y \in \mathbb{R}^n, \forall \lambda \in [0, 1], \qquad f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

Convexity and differentiability

Proposition

A differentiable function f is convex iff

$$\forall x, \bar{x} \in \operatorname{dom} f, \quad f(x) \geq f(\bar{x}) + \langle \nabla f(\bar{x}), x - \bar{x} \rangle$$



Proposition

A differentiable function f is convex iff

$$\forall x, \bar{x} \in \operatorname{dom} f, \quad f(x) \geq f(\bar{x}) + \langle \nabla f(\bar{x}), x - \bar{x} \rangle$$

Proof idea

- (Convex $\implies$ tangent). Divide by λ the relation

$$f(\lambda x + (1-\lambda)\bar{x}) - f(\bar{x}) \leq \lambda(f(x) - f(\bar{x})).$$

- (Tangent $\implies$ convex). Apply the tangent relation twice to $z = \lambda x + (1-\lambda)y$ and x (resp. y), and use a convex combination of the two relationships.