Farthest distance function to strongly convex sets in Hilbert spaces

Nguyen Vo Anh Thuong

Based on a joint work with:
Florent Nacry (Université de Perpignan Via Domitia)
Lionel Thibault (Université de Montpellier)

Université de Perpignan Via Domitia
LAboratoire de Modélisation Plurisdisciplinaire et Simulations

October 12, 2022

Content

1 Overview
- Notation
- Prox-regularity
- Strongly convexity

2 Semiconcavity of the farthest distance function
- Previous results
- Semiconcavity property

3 Farthest distance and separating balls
- Separation property
- Some analytic formulation for the farthest distance function to a strongly convex set

Overview

- The letter X stands for (real) Hilbert spaces endowed with an inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$.
- For every $\emptyset \neq S \subset X$ and every $x \in X$, The distance function d_S and the farthest distance function dfar_S (*resp.*, the nearest points and the farthest points) from S to x are defined by

$$d_S(x) := d(x, S) := \inf_{y \in S} \|x - y\| \text{ and}$$

$$\mathrm{dfar}_S(x) := \mathrm{dfar}(x, S) := \sup_{y \in S} \|x - y\|$$

$$\Big(\textit{resp.,}\quad \mathrm{Proj}_S(x) := \mathrm{Proj}(S, x) := \{y \in S : d_S(x) = \|x - y\|\} \text{ and}$$

$$\mathrm{Far}_S(x) := \mathrm{Far}(S, x) := \{y \in S : \mathrm{dfar}_S(x) = \|x - y\|\}\Big).$$

When $\mathrm{Proj}_S(x)$ (*resp.*, $\mathrm{Far}_S(x)$) contains one and only one vector $y \in X$, we set $\mathrm{proj}_S(x) := y$ (*resp.*, $\mathrm{far}_S(x)$).

• Let C be a convex set and let $x \in C$. The **normal cone** in the sense of convex analysis to x in C is the set of normal vector y to x such that

$$N_C(x) = \{y \in \mathcal{H} :< y, z - x > \leq 0, \forall z \in C\}.$$

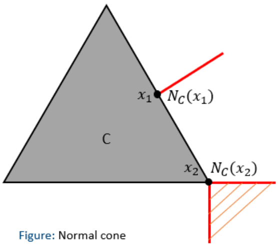


Figure: Normal cone
In sense of convex analysis

• Let C be a convex set and let $x \in C$. The **normal cone** in the sense of convex analysis to x in C is the set of normal vector y to x such that

$$N_C(x) = \{y \in \mathcal{H} :< y, z - x >\leq 0, \forall z \in C\}.$$

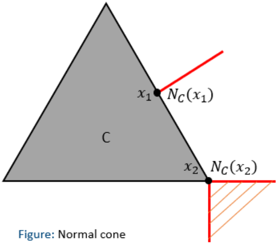


Figure: Normal cone
In sense of convex analysis

• Let S be a nonempty subset in X. The **proximal normal cone** to S at $x \in S$ is

$$N^P(S; x) := \{v \in \mathcal{H} : \exists\, r > 0,\ x \in \mathrm{Proj}_S(x + rv)\}.$$

• Let C be a convex set and let $x \in C$. The **normal cone** in the sense of convex analysis to x in C is the set of normal vector y to x such that

$$N_C(x) = \{y \in \mathcal{H} : < y, z - x > \leq 0, \forall z \in C\}.$$

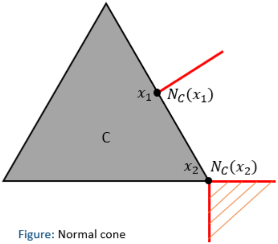


Figure: Normal cone
In sense of convex analysis

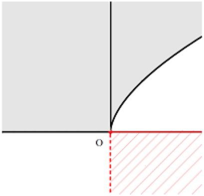


Figure: Proximal normal cone

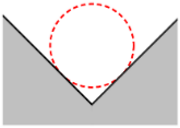


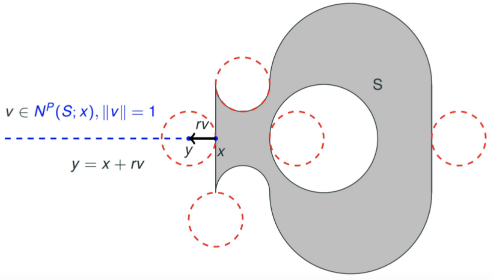
Figure: N^P is often reduced to 0

• Let S be a nonempty subset in X. The **proximal normal cone** to S at $x \in S$ is

$$N^P(S; x) := \{v \in \mathcal{H} : \exists\, r > 0,\ x \in \mathrm{Proj}_S(x + rv)\}.$$

- Let S be a nonempty subset in X. The **proximal normal cone** to S at $x \in S$ is

$$N^P(S;x) := \{v \in \mathcal{H} : \exists\, r > 0,\ x \in \mathrm{Proj}_S(x + rv)\}.$$



Definition

Let S be a nonempty subset of $\mathcal{H}$ and $r \in]0, +\infty]$. One say that S is **r-prox-regular** whenever, for all $x \in S$ and for all $v \in N^P(S, x) \cap \mathbb{B}$, one has

$$x \in \mathrm{Proj}_S(x + tv) \text{ for any real } t \in]0, r].$$

or equivalently $$B(x + tv, t) \cap S = \emptyset.$$

Definition

Let S be a nonempty subset of $\mathcal{H}$ and $r \in]0, +\infty]$. One say that S is **r-prox-regular** whenever, for all $x \in S$ and for all $v \in N^P(S, x) \cap \mathbb{B}$, one has

$$x \in \mathrm{Proj}_S(x + tv) \text{ for any real } t \in]0, r].$$

or equivalently $$B(x + tv, t) \cap S = \emptyset.$$

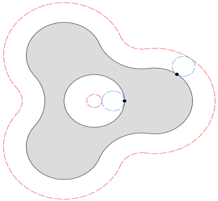


Figure: Prox-regular set

Definition

Let S be a nonempty subset of $\mathcal{H}$ and $r \in]0,+\infty]$. One say that S is **r-prox-regular** whenever, for all $x \in S$ and for all $v \in N^P(S,x) \cap \mathbb{B}$, one has

$$x \in \mathrm{Proj}_S(x+tv) \text{ for any real } t \in]0,r].$$

or equivalently $$B(x+tv,t) \cap S = \emptyset.$$

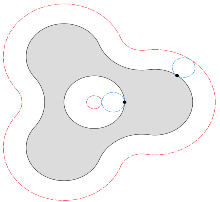


Figure: Prox-regular set

The set S is r**-prox-regular** whenever $\mathrm{proj}_S : U_r(S) := \{x \in \mathcal{H} : d_S(x) < r\} \to X$ is well-defined and norm-to-norm continuous.

Definition

Let S be a nonempty subset of $\mathcal{H}$ and $r \in]0, +\infty]$. One say that S is **r-prox-regular** whenever, for all $x \in S$ and for all $v \in N^P(S, x) \cap \mathbb{B}$, one has

$$x \in \mathrm{Proj}_S(x + tv) \text{ for any real } t \in]0, r].$$

or equivalently
$$B(x + tv, t) \cap S = \emptyset.$$

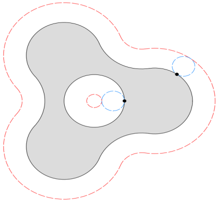


Figure: Prox-regular set

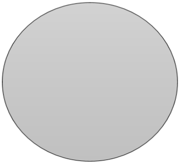


Figure: Nonempty closed convex $\Leftrightarrow$ ∞-prox-regular

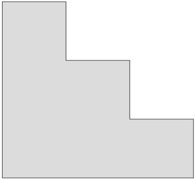


Figure: Lack of prox-regularity

Definition

Let S be a nonempty subset of $\mathcal{H}$ and $r \in]0, +\infty]$. One say that S is **r-prox-regular** whenever, for all $x \in S$ and for all $v \in N^P(S, x) \cap \mathbb{B}$, one has

$$x \in \mathrm{Proj}_S(x + tv) \text{ for any real } t \in]0, r].$$

or equivalently $$B(x + tv, t) \cap S = \emptyset.$$

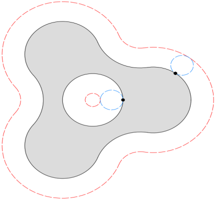


Figure: Prox-regular set

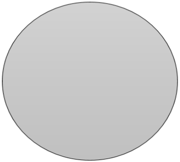


Figure: Nonempty closed convex $\Leftrightarrow$ ∞-prox-regular

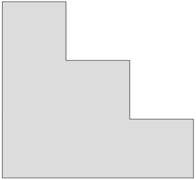


Figure: Lack of prox-regularity

Contributors: H. Federer (1959); J.-P. Vial (1983); A. Canino (1988); A.Shapiro (1994); F.H. Clarke, R.L. Stern, P.R. Wolenski (1995); R.A. Poliquin, R.T. Rockafellar, L. Thibault (2000).

Theorem ([3])

Let S be a nonempty closed subset of $\mathcal{H}$ and $r > 0$. **The following are equivalent:**

(a) *S is r-prox-regular;*

(b) *for all $x, x' \in S$, for all $v \in N^P(S; x)$, one has*

$$\langle v, x' - x\rangle \leq \frac{1}{2r} \|v\| \left\|x - x'\right\|^2;$$

(c) *the mapping $\mathrm{proj}_S(\cdot)$ is well defined on $U_r(S)$, and for every real $s \in]0, r[$, for all $x, x' \in U_s(S)$,*

$$\left\|\mathrm{proj}_S(x) - \mathrm{proj}_S(x')\right\| \leq \frac{1}{1 - (s/r)} \left\|x - x'\right\|;$$

(d) *for any $u \in U_r(S) \setminus S$ such that $\mathrm{proj}_S(u) =: x$ is well defined, one has*

$$x = \mathrm{proj}_S\left(x + \frac{t}{d_S(u)}(u - x)\right) \quad \text{for all } t \in [0, r[;$$

(e) *the function $d_S^2(\cdot)$ is differentiable on $U_r(S)$ with a locally Lipschitz derivative and*

$$\nabla d_S^2(x) = 2\left(x - \mathrm{proj}_S(x)\right) \quad \text{for all } x \in U_r(S).$$

Theorem ([3])

Let S be a nonempty closed subset of $\mathcal{H}$ and $r > 0$. **The following are equivalent:**
(a) S is r-prox-regular;
(b) for all $x, x' \in S$, for all $v \in N^P(S;x)$, one has

$$\langle v, x' - x\rangle \leq \frac{1}{2r}\|v\|\left\|x - x'\right\|^2;$$

(c) the mapping $\mathrm{proj}_S(\cdot)$ is well defined on $U_r(S)$, and for every real $s \in]0, r[$, for all $x, x' \in U_s(S)$,

$$\left\|\mathrm{proj}_S(x) - \mathrm{proj}_S(x')\right\| \leq \frac{1}{1 - (s/r)}\left\|x - x'\right\|;$$

(d) for any $u \in U_r(S) \setminus S$ such that $\mathrm{proj}_S(u) =: x$ is well defined, one has

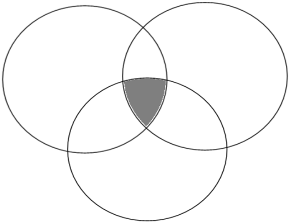
$$x = \mathrm{proj}_S\left(x + \frac{t}{d_S(u)}(u - x)\right) \quad \text{for all } t \in [0, r[;$$

(e) the function $d_S^2(\cdot)$ is differentiable on $U_r(S)$ with a locally Lipschitz derivative and

$$\nabla d_S^2(x) = 2\left(x - \mathrm{proj}_S(x)\right) \quad \text{for all } x \in U_r(S).$$

- If S is r-prox-regular, then Fréchet N^F, Mordukhovich limiting N^L, Clarke N^C)

$$N^P(S;\cdot) = N^F(S;\cdot) = N^L(S;\cdot) = N^C(S;\cdot) := N(S;\cdot).$$



Definition

Let C be a nonempty subset in X. One says that C is R**-strongly convex** for some real $R > 0$ whenever there is a nonempty set $L \subset X$ such that

$$C = \bigcap_{x \in L} B[x, R].$$

Contributors: H. Frankowska, C. Olech (1981), J. P. Vial (1985), E. S. Polovinkin (1996, 2000), M. V. Balashov (2004, 2006), G. E. Ivanov (1995, 2006), A. Weber and G. Reibig (2013).

Theorem ([7, 10])

Let C be a nonempty closed convex bounded subset in X and let $R > 0$ be a real. The following assertions are equivalent:
(a) the set C is R-strongly convex;
(b) for all $x, x' \in C$ and for all $v \in N(C; x)$, one has

$$\langle v, x' - x \rangle \leq -\frac{\|v\|}{2R} \|x' - x\|^2;$$

(c) the mapping far_C is well defined on $\mathcal{E}_R(C) := \{x \in X : \mathrm{dfar}_C(x) > R\}$ and for every real $s > R$, for all $x, x' \in \mathcal{E}_s(C)$,

$$\|\mathrm{far}_C(x) - \mathrm{far}_C(x')\| \leq \frac{1}{(s/R) - 1} \|x - x'\|;$$

(d) for any $u \in \mathcal{E}_R(C)$ such that $\mathrm{far}_C(u) =: x$ is well defined, one has

$$x = \mathrm{far}_C\big(x - \frac{t}{\mathrm{dfar}_C(u)}(x - u)\big) \quad \text{for all } t \in]R, +\infty[;$$

(e) the function $\mathrm{dfar}^2_C(\cdot)$ is differentiable on $\mathcal{E}_R(C)$ with a locally Lipschitz derivative and

$$\nabla \mathrm{dfar}^2_C(x) = 2\big(x - \mathrm{far}_C(x)\big) \quad \text{for all } x \in \mathcal{E}_R(C).$$

Semiconcavity of the farthest distance function

The function d_C is convex if and only if the set C is convex.

The function d_C is convex if and only if the set C is convex.

$\Rightarrow$ **Q1:** Can we characterize the set for which its distance is semiconvex?

The function d_C is convex if and only if the set C is convex.

$\Rightarrow$ **Q1:** Can we characterize the set for which its distance is semiconvex?

Definition

Let U a nonempty convex subset of X and $\sigma \geq 0$ be a real. A function $f : U \to \mathbb{R} \cup \{+\infty\}$ is said to be linearly σ-*semiconvex* on U provided that for every $t \in]0,1[$ and every $x, y \in U$

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y) + \frac{\sigma}{2} t(1-t) \|x - y\|^2$$

or equivalently if $f(\cdot) + \sigma/2 \|\cdot\|^2$ is convex on U.

The function d_C is convex if and only if the set C is convex.

$\Rightarrow$ **Q1:** Can we characterize the set for which its distance is semiconvex?

Definition

Let U a nonempty convex subset of X and $\sigma \geq 0$ be a real. A function $f : U \to \mathbb{R} \cup \{+\infty\}$ is said to be linearly σ-*semiconvex* on U provided that for every $t \in]0, 1[$ and every $x, y \in U$

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y) + \frac{\sigma}{2}t(1-t)\|x-y\|^2$$

or equivalently if $f(\cdot) + \sigma/2\|\cdot\|^2$ is convex on U.

Let S be an r-prox-regular.

- *G. Colombo and L. Thibault* ([3]): It is equivalent to for any real $0 < s < r$, the function d_S^2 is $s/(r-s)$-semiconvex on any convex set included in $U_s(S)$.
- *M. V. Balashov* ([6]): The function d_S is $(r-s)^{-1}$- semiconvex on any convex set included in $U_s(S)$.
- *F. Nacry and L. Thibault* ([4]): Provide a short proof of Balashov's result by establishing that a prox-regular set is nothing but the complement of union of closed balls of common radius.

Q2: Can we replace the usual distance function by the farthest distance?

Q2: Can we replace the usual distance function by the farthest distance?

Theorem (Balashov-Golubev ([6], 2014), Nacry-N.V.A.T-Thibault (2022))

Let C be a nonempty closed bounded subset of X and let $R > 0$ be a positive real. The following assertions are equivalent:
(a) the set C is R-strongly convex;

Q2: Can we replace the usual distance function by the farthest distance?

Theorem (Balashov-Golubev ([6], 2014), Nacry-N.V.A.T-Thibault (2022))

Let C be a nonempty closed bounded subset of X and let $R > 0$ be a positive real. The following assertions are equivalent:
(a) the set C is R-strongly convex;
(b) for any real $s > R$, the function $-\mathrm{dfar}_C$ is linearly semiconvex on nonempty convex subset V of $\mathcal{E}_R(C)$ with $(s-R)^{-1}$ as coefficient;

Q2: Can we replace the usual distance function by the farthest distance?

Theorem (Balashov-Golubev ([6], 2014), Nacry-N.V.A.T-Thibault (2022))

Let C be a nonempty closed bounded subset of X and let $R > 0$ be a positive real. The following assertions are equivalent:
(a) the set C is R-strongly convex;
(b) for any real $s > R$, the function $-\mathrm{dfar}_C$ is linearly semiconvex on nonempty convex subset V of $\mathcal{E}_R(C)$ with $(s-R)^{-1}$ as coefficient;
(c) the function $-\mathrm{dfar}_C$ is locally linearly semiconvex on $\mathcal{E}_R(C)$, that is, linearly semiconvex near each point in $\mathcal{E}_R(C)$.

Main idea of the proof: (a $\Rightarrow$ b)

- Let C be R-strongly convex with $R > 0$ and let S be r-prox-regular with $r > 0$ such that $0 < R < r$. The set $C + S$ is $(r - R)$-prox-regular, so closed ([10]).
- Let S be an r-prox-regular with $r > 0$. Then, for all $s \in]0, r[$, the set $X \setminus S$ is the union of a family of closed balls of X of radius s ([4]).
- If S is the union of a collection of closed balls with a $r > 0$, then on each nonempty convex set U included in $cl(X \setminus S)$, then d_S is r^{-1}-semiconcave ([8]).

Main idea of the proof: (a $\Rightarrow$ b)

- Let C be R-strongly convex with $R > 0$ and let S be r-prox-regular with $r > 0$ such that $0 < R < r$. The set $C + S$ is $(r - R)$-prox-regular, so closed ([10]).

  $\Rightarrow$ The set $D := \{\mathrm{dfar}_C \geq s\}$ is $(s - R)$-prox-regular.

- Let S be an r-prox-regular with $r > 0$. Then, for all $s \in]0, r[$, the set $X \setminus S$ is the union of a family of closed balls of X of radius s ([4]).

- If S is the union of a collection of closed balls with a $r > 0$, then on each nonempty convex set U included in $cl(X \setminus S)$, then d_S is r^{-1}-semiconcave ([8]).

Main idea of the proof: (a $\Rightarrow$ b)

- Let C be R-strongly convex with $R > 0$ and let S be r-prox-regular with $r > 0$ such that $0 < R < r$. The set $C + S$ is $(r - R)$-prox-regular, so closed ([10]).

  $\Rightarrow$ The set $D := \{\mathrm{dfar}_C \geq s\}$ is $(s - R)$-prox-regular.

- Let S be an r-prox-regular with $r > 0$. Then, for all $s \in]0, r[$, the set $X \setminus S$ is the union of a family of closed balls of X of radius s ([4]).

  $\Rightarrow$ $S := X \setminus D = \{\mathrm{dfar}_C < s\}$ is the union of a collection of closed balls with radius $s - t$ for any $t \in]R, s[$.

- If S is the union of a collection of closed balls with a $r > 0$, then on each nonempty convex set U included in $cl(X \setminus S)$, then d_S is r^{-1}-semiconcave ([8]).

Main idea of the proof: (a $\Rightarrow$ b)

- Let C be R-strongly convex with $R > 0$ and let S be r-prox-regular with $r > 0$ such that $0 < R < r$. The set $C + S$ is $(r - R)$-prox-regular, so closed ([10]).

  $\Rightarrow$ The set $D := \{\mathrm{dfar}_C \geq s\}$ is $(s - R)$-prox-regular.

- Let S be an r-prox-regular with $r > 0$. Then, for all $s \in]0, r[$, the set $X \setminus S$ is the union of a family of closed balls of X of radius s ([4]).

  $\Rightarrow$ $S := X \setminus D = \{\mathrm{dfar}_C < s\}$ is the union of a collection of closed balls with radius $s - t$ for any $t \in]R, s[$.

- If S is the union of a collection of closed balls with a $r > 0$, then on each nonempty convex set U included in $cl(X \setminus S)$, then d_S is r^{-1}-semiconcave ([8]).

  $\Rightarrow$ $d(\cdot, S)$ is $(s - t)^{-1}$-linearly semiconcave on $U \subset D$.

Main idea of the proof: (a $\Rightarrow$ b)

- Let C be R-strongly convex with $R > 0$ and let S be r-prox-regular with $r > 0$ such that $0 < R < r$. The set $C + S$ is $(r - R)$-prox-regular, so closed ([10]).

  $\Rightarrow$ The set $D := \{\mathrm{dfar}_C \geq s\}$ is $(s - R)$-prox-regular.

- Let S be an r-prox-regular with $r > 0$. Then, for all $s \in]0, r[$, the set $X \setminus S$ is the union of a family of closed balls of X of radius s ([4]).

  $\Rightarrow$ $S := X \setminus D = \{\mathrm{dfar}_C < s\}$ is the union of a collection of closed balls with radius $s - t$ for any $t \in]R, s[$.

- If S is the union of a collection of closed balls with a $r > 0$, then on each nonempty convex set U included in $cl(X \setminus S)$, then d_S is r^{-1}-semiconcave ([8]).

  $\Rightarrow$ $d(\cdot, S)$ is $(s - t)^{-1}$-linearly semiconcave on $U \subset D$.

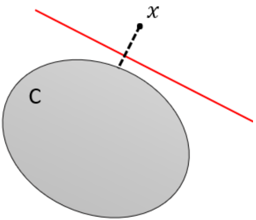
- Also prove that: $\mathrm{dfar}_C(u) = s + d\big(u, \{\mathrm{dfar}_C \leq s\}\big)$ for all $u \in \mathcal{E}_s(C)$.

$$\boxed{\mathrm{dfar}_C(\cdot) \text{ is } (s - t)^{-1}\text{-linearly semiconcave.}}$$

Farthest distance and separating balls

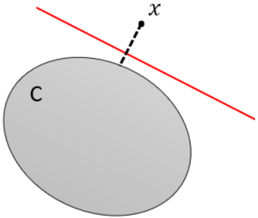
- Let C be a nonempty closed convex set of X and $x \in X \setminus C$, it is well-known that with $x_\star := d_C(x)^{-1}\big(x - \mathrm{proj}_C(x)\big)$, the separation property for some real α

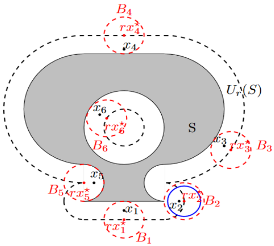
$$C \subset \{\langle x_\star, \cdot\rangle < \alpha\} \quad \text{and} \quad \langle x_\star, x\rangle > \alpha.$$



- Let C be a nonempty closed convex set of X and $x \in X \setminus C$, it is well-known that with $x_\star := d_C(x)^{-1}\big(x - \mathrm{proj}_C(x)\big)$, the separation property for some real α

$$C \subset \{\langle x_\star, \cdot\rangle < \alpha\} \quad \text{and} \quad \langle x_\star, x\rangle > \alpha.$$





- Replacing $\{\langle x_\star, \cdot\rangle < \alpha\}$ by a general form $\left\{\langle x_\star, x\rangle - \frac{\|x\|^2}{2r} < \alpha\right\}$ allows to extend the separation property, to r-prox-regular sets.

Proposition ([1])

Let S be an r-prox-regular subset of X with $r > 0$, $x \in X$ with $\delta := d_S(x) \in]0, r[$. Then, with $x_\star := (\frac{1}{r} - \frac{1}{\delta})\mathrm{proj}_S(x) + \frac{1}{\delta}x$ the separation property for some $\alpha \in \mathbb{R}$

$$S \subset \left\{\langle x_\star, \cdot\rangle - \frac{\|\cdot\|^2}{2r} < \alpha\right\} \quad \text{and} \quad \langle x_\star, x\rangle - \frac{\|x\|^2}{2r} > \alpha.$$

- **Aim**: provide the separation property in the case of strongly convex sets.

▶ **Aim**: provide the separation property in the case of strongly convex sets.

Given any $x^\star \in X$, any real $R > 0$ and any R-strongly convex set C in X, set

$$q_{x^\star,R}(x) := \langle x^\star, x\rangle - \frac{\|x\|^2}{2R} \quad \text{for all } x \in X$$

and

$$\Upsilon_{C,R}(x) := \Big(\frac{1}{R} - \frac{1}{\mathrm{dfar}_C(x)}\Big)\mathrm{far}_C(x) + \frac{1}{\mathrm{dfar}_C(x)}x \quad \text{for all } x \in \mathcal{E}_R(C).$$

▶ **Aim**: provide the separation property in the case of strongly convex sets.

Given any $x^\star \in X$, any real $R > 0$ and any R-strongly convex set C in X, set

$$q_{x^\star,R}(x) := \langle x^\star, x\rangle - \frac{\|x\|^2}{2R} \quad \text{for all } x \in X$$

and

$$\Upsilon_{C,R}(x) := \Big(\frac{1}{R} - \frac{1}{\mathrm{dfar}_C(x)}\Big)\mathrm{far}_C(x) + \frac{1}{\mathrm{dfar}_C(x)} x \quad \text{for all } x \in \mathcal{E}_R(C).$$

Theorem

Let C be an R-strongly convex set in X for some real $R > 0$ and let $x \in X$ with $\delta := \mathrm{dfar}_C(x) > 2R$. Then, with $x^\star = \Upsilon_{C,R}(x)$, for some $\alpha \in \mathbb{R}$,

$$C \subset \left\{\langle x^\star, \cdot\rangle - \frac{\|\cdot\|^2}{2R} > \alpha\right\} \quad \text{and} \quad q_{x^\star,R}(x) < \alpha \le \inf_{c\in C} q_{x^\star,R}(c).$$

The farthest distance in terms of **support function**

- Let C be a closed convex set in X, the distance function d_C has been described in terms of the *support function* $\sigma(\cdot, C)$ at $x_\star := d_C(x)^{-1}(x - \mathrm{proj}_C(x))$

$$d_C(x) = \langle x_\star, x\rangle - \sigma(x_\star, C).$$

- It has been extended to the context of a prox-regular set (in [4]), that means for an r-prox-regular set S with $r > 0$, one has

$$d_S(x)\left(1 - \frac{d_S(x)}{2r}\right) = q_{x_\star,r}(x) - \phi_{S,r}(x_\star),$$

where $\phi_{S,r}(x_\star) := \sup_{u\in S} q_{x_\star,r}(u)$ with $q_{x_\star,r}(x) := \langle x_\star, x\rangle - \frac{\|x\|^2}{2r}$ for all $x \in X$.

The farthest distance in terms of **support function**

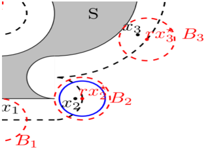
- Let C be a closed convex set in X, the distance function d_C has been described in terms of the *support function* $\sigma(\cdot, C)$ at $x_\star := d_C(x)^{-1}(x - \mathrm{proj}_C(x))$
$$d_C(x) = \langle x_\star, x\rangle - \sigma(x_\star, C).$$
- It has been extended to the context of a prox-regular set (in [4]), that means for an r-prox-regular set S with $r > 0$, one has
$$d_S(x)\left(1 - \frac{d_S(x)}{2r}\right) = q_{x_\star,r}(x) - \phi_{S,r}(x_\star),$$
where $\phi_{S,r}(x_\star) := \sup_{u\in S} q_{x_\star,r}(u)$ with $q_{x_\star,r}(x) := \langle x_\star, x\rangle - \frac{\|x\|^2}{2r}$ for all $x \in X$.
- **Aim**: provide the formula of farthest distance in terms of support function in the case of strongly convex sets.

Lemma

Let C be an R-strongly convex subset of X for some real $R > 0$ and let $x \in X$ with $\mathrm{dfar}_C(x) > R$. Then, there exists one and only one $x^\star \in X$ with $\|x^\star - R^{-1}x\| = R^{-1}\mathrm{dfar}_C(x) - 1$ (namely, $x^\star := \Upsilon_{C,R}(x)$) such that
$$\mathrm{dfar}_C(x)\left(1 - \frac{\mathrm{dfar}_C(x)}{2R}\right) = q_{x^\star,R}(x) - \Phi_{C,R}(x^\star)$$
where $\Phi_{C,R}(x^\star) := \inf_{c\in C} q_{x^\star,R}(c)$.

The farthest distance in terms of **supporting hyperplanes**

- In ([2]): the distance of a point to prox-regular set is the maximum of the distances of the point from boundaries of all such complements separating the set and the point.



The farthest distance in terms of **supporting hyperplanes**

- In ([2]): the distance of a point to prox-regular set is the maximum of the distances of the point from boundaries of all such complements separating the set and the point.

$\Rightarrow$ Develop in case of strongly convex set.

The farthest distance in terms of **supporting hyperplanes**

- In ([2]): the distance of a point to prox-regular set is the maximum of the distances of the point from boundaries of all such complements separating the set and the point.

$\Rightarrow$ Develop in case of strongly convex set.

$$\text{Setting} \quad \Phi_{C,R}(x^\star) := \inf_{c\in C} q_{x^\star,R}(c) \quad \text{and} \quad L_{x^\star,R,\alpha} := \{q_{x^\star,R} \geq \alpha\}$$

Theorem

Let C be an R-strongly subset of X for some $R > 0$ and let $x \in X$ with $\delta := \mathrm{dfar}(x, C) > 2R$. One has

$$\delta = \min\left\{\mathrm{dfar}(x, L_{y^\star,R,\alpha}) : (y^\star,\alpha) \in X \times \mathbb{R}, C \subset L_{y^\star,R,\alpha}, x \notin L_{y^\star,R,\alpha}\right\}.$$

The minimum is attained at $(x^\star, \beta)$ with $x^\star := \Upsilon_{C,R}(x)$ and $\beta := \Phi_{C,R}(x^\star)$. Further, for all $y^\star \in X$ with $\left\|y^\star - R^{-1}x\right\| = R^{-1}\delta - 1$ and all $\alpha \in \mathbb{R}$, one has

$$\left.\begin{array}{l}\delta = d(x, L_{y^\star,r,\alpha}),\\ C \subset L_{y^\star,r,\alpha}, x \notin L_{y^\star,r,\alpha}\end{array}\right\} \Rightarrow (y^\star,\alpha) = \left(x^\star, \Phi_{C,R}(x^\star)\right).$$

Conclusion: In this paper,

1. Develop some properties of strongly convex sets through the farthest distance function, especially semiconvavity.
2. Provide the separation to strongly convex sets from an outside point in Hilbert space.

Perspective the research

1. Involving strong convexity to some aspects of Variational Analysis.
2. Involving strong convexity in differential inclusion (Sweeping process theory).
3. Consider metric regularity of $d(C \cap S; x)$.
4. Study in Banach spaces.

- S. Adly, F. Nacry, L. Thibault, *Prox-regular sets and Legendre-Fenchel transform related to separation properties.* Optimization 71 (2022), 2097-2129.
- S. Adly, F. Nacry, L. Thibault, *New metric properties for prox-regular sets.* Math. Program. 189 (2021), Ser. B, 7-36.
- G. Colombo, L. Thibault, *Prox-regular sets and applications.* Handbook of nonconvex analysis and applications, 99-182 (2010), Int. Press, Somerville, MA.
- F. Nacry, L. Thibault, *Distance function associated to a prox-regular set.* Set-Valued Var. Anal 30 (2022), 731-750.
- M. V. Balashov, *Weak convexity of the distance function.* J. Convex Anal, 20 (2013), 93-106.
- M. V. Balashov, M. O. Golubev, *Weak concavity of the antidistance function.* J. Convex Anal. 21 (2014), 951-964.
- M. V. Balashov, G. E. Ivanov, *On farthest points of sets.* Mathematical Notes, 80 (2006), 159-166.
- P. Cannarsa, C. Sinestrari, *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control.* Birkhäuser Boston, Inc., Boston, MA, 2004.
- F. Deutsch, *Best Approximation in Inner Product Spaces.* CMS Books in Mathematics, 7, Springer-Verlag, New-York, 2001.
- G. E. Ivanov, *Weakly Convex Sets and Functions: Theory and Applications.* Fizmatlit, Moscow (2006) (in Russian).