Optimal location of resources for species survival

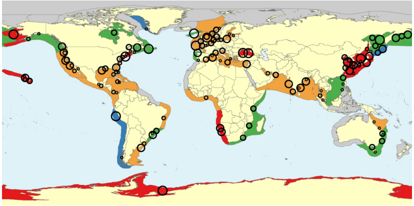
Yannick Privat

Univ. Strasbourg, IRMA

oct. 2022

Outline

1. Modeling issues : toward a shape optimization problem
2. Analysis of optimal resources domains
   - Known results about the minimizers of $\lambda(m)$
   - New results on $\lambda(m)$: a Faber-Krahn type inequality ?
   - Maximizing the total population size
3. Biased movement of species
4. Towards a more concrete problem



J. Lamboley, A. Laurain, G. Nadin, Y. Privat, *Properties of optimizers of the principal eigenvalue with indefinite weight and Robin conditions*, Calc. Var. Partial Differential Equations 55 (2016), no. 6.

I. Mazari, G. Nadin, Y. Privat, *Optimal location of resources maximizing the total population size in logistic models*, Journal Math. Pures Appl. (9) 134 (2020), 1–35.

Outline

Biological model : population dynamics

Logistic diffusive equation (Fisher-Kolmogorov 1937, Fleming 1975, Cantrell-Cosner 1989)

Introduce

$\rightsquigarrow$ $\Omega \subset \mathbb{R}^N$: bounded domain with Lipschitz boundary (habitat)

$\rightsquigarrow$ μ : diffusion coefficient ($\mu > 0$)

$\rightsquigarrow$ $u(t, x)$: density of a species at location x and time t

$\rightsquigarrow$ $m(x)$: control - intrinsic growth rate of species at location x and

- $\Omega \cap \{m > 0\}$ (resp. $\Omega \cap \{m < 0\}$) is the favorable (resp. unfavorable) part of habitat
- $\int_\Omega m$ measures the total resources in the spatially heterogeneous environment Ω
- After renormalization, one is allowed to assume that

$$-1 \leq m(x) \leq \kappa \quad \text{with } \kappa > 0 \quad \text{and } m \text{ changes sign.}$$

Biological model

$$\begin{cases} u_t = \mu \Delta u + u[m(x) - u] & \text{in } \Omega \times \mathbb{R}_+, \\ u(0, x) \geq 0, \quad u(0, x) \not\equiv 0 & \text{in } \overline{\Omega}, \end{cases}$$

Biological model : population dynamics

Choice of boundary conditions

$$\partial_n u = 0 \qquad \text{on } \partial\Omega \times \mathbb{R}^+ \quad \text{(no-flux boundary condition)}$$

Here, the boundary $\partial\Omega$ acts as a barrier
$\rightsquigarrow$ other kinds of B.C. have been considered in this study

The complete model

$$\left\{\begin{array}{ll} u_t = \mu\Delta u + u[m(x) - u] & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_n u = 0 & \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0,x) \geq 0, \quad u(0,x) \not\equiv 0 & \text{in } \overline{\Omega}, \end{array}\right.$$

($\rightsquigarrow$ takes into account effects of dispersal and partial heterogeneity)

Analysis of the model : extinction/survival condition

The complete model

$$\begin{cases} u_t = \mu \Delta u + u[m(x) - u] & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_n u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0,x) \geq 0, \quad u(0,x) \not\equiv 0 & \text{in } \overline{\Omega}, \end{cases}$$

Introduce the eigenvalue problem

$$\boxed{\begin{cases} \Delta\varphi + \lambda m \varphi = 0 & \text{in } \Omega, \\ \partial_n \varphi = 0 & \text{on } \partial\Omega, \end{cases}} \qquad (EP)$$

Existence of a positive principal eigenvalue $\lambda(m)$

- if $\int_\Omega m < 0$, then (EP) has a unique principal eigenvalue $\lambda(m)$.
- if $\int_\Omega m \geq 0$, then 0 is the unique nonnegative principal eigenvalue of (EP).

Analysis of the model : extinction/survival condition

The complete model

$$\begin{cases} u_t = \mu\Delta u + u[m(x) - u] & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_n u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0,x) \geq 0, \quad u(0,x) \not\equiv 0 & \text{in } \overline{\Omega}, \end{cases}$$

Introduce the eigenvalue problem

$$\boxed{\begin{cases} \Delta\varphi + \lambda m\varphi = 0 & \text{in } \Omega, \\ \partial_n \varphi = 0 & \text{on } \partial\Omega, \end{cases}} \qquad (EP)$$

Theorem (Cantrell-Cosner 1989, Berestycki-Hamel-Roques 2005)

Let u^* be the unique positive steady solution of the logistic equation above. One has

- $\mu \geq 1/\lambda(m) \quad \Longrightarrow \quad u(t,x) \underset{t\to\infty}{\longrightarrow} 0,$
- $\mu < 1/\lambda(m) \quad \Longrightarrow \quad u(t,x) \underset{t\to\infty}{\longrightarrow} u^*(x).$

Comments on the eigenvalue problem (with a sign changing weight m)

Characterization of $\lambda(m)$

$\lambda(m)$ is the unique principal ($\Leftrightarrow \varphi > 0$) positive eigenvalue of the problem :

$$\begin{cases} \Delta\varphi + \lambda m\varphi = 0 & \text{in } \Omega, \\ \partial_n\varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

Another characterization of $\lambda(m)$

$\lambda(m)$ is also characterized by the min-formula :

$$\lambda(m) = \inf\left\{ \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega m\varphi^2}, \quad \varphi \in H^1(\Omega), \ \int_\Omega m\varphi^2 > 0 \right\}.$$

Optimal arrangements of resources

Conclusion of this part : 2 optimal control problems

$$\boxed{u_t = \mu\Delta u + u[m(x) - u]}$$

Dynamical problem	Static problem
$\boxed{\Delta\varphi + \lambda m\varphi = 0}$	$\boxed{\mu\Delta u^* + u^*(m - u^*) = 0}$
$\rightsquigarrow$ species can be maintained iff $\mu < 1/\lambda(m)$. Hence, the smaller $\lambda(m)$ is, the more likely the species can survive	$\rightsquigarrow$ maximizes the total size of the population
$\boxed{\inf_{m\in\mathcal{M}_{m_0,\kappa}} \lambda(m)} \qquad (P_{\text{Dyn}})$	$\boxed{\sup_{m\in\mathcal{M}_{m_0,\kappa}} \int_\Omega u^*} \qquad (P_{\text{Stat}})$

Optimal arrangements of resources

Conclusion of this part : 2 optimal control problems

$$\boxed{u_t = \mu\Delta u + u[m(x) - u]}$$

Dynamical problem	Static problem
$\boxed{\Delta\varphi + \lambda m\varphi = 0}$	$\boxed{\mu\Delta u^* + u^*(m - u^*) = 0}$
$\rightsquigarrow$ species can be maintained iff $\mu < 1/\lambda(m)$. Hence, the smaller $\lambda(m)$ is, the more likely the species can survive	$\rightsquigarrow$ maximizes the total size of the population
$\boxed{\inf_{m\in\mathcal{M}_{m_0,\kappa}} \lambda(m)}$ (P_{Dyn})	$\boxed{\sup_{m\in\mathcal{M}_{m_0,\kappa}} \int_\Omega u^*}$ (P_{Stat})

Choice of admissible weights

$$\mathcal{M}_{m_0,\kappa} = \left\{ m \in L^\infty(\Omega, [-1,\kappa]),\ |\{m > 0\}| > 0,\ \int_\Omega m \leq -m_0|\Omega| \right\}$$

Outline

Outline

Bang-bang property of minimizers

Proposition (Lou-Yanagida 2006, Derlet-Gossez-Takac 2010)

Problem (P_{Dyn}) has a solution. Moreover, every minimizer m satisfies

$$\int_\Omega m = -m_0|\Omega| \qquad \text{and} \qquad m = \kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}.$$

- Easy : $m \mapsto \lambda(m)$ is continuous for the L^∞ weak-$\star$ topology and the set of admissible weights $\mathcal{M}_{m_0,\kappa}$ is compact.
- Direct computations show that

$$\begin{aligned}\lambda(m) &= \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega m\varphi^2} \\ &\geq \frac{\int_\Omega |\nabla\varphi|^2}{\sup_{\tilde m \in \mathcal{M}_{m_0,\kappa}} \int_\Omega \tilde m \varphi^2} = \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega (\kappa\mathbb{1}_E - \mathbb{1}_{\Omega\setminus E})\varphi^2} \geq \lambda(\kappa\mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}),\end{aligned}$$

  where E is chosen in such a way that

$$\{\varphi > t\} \subset E \subset \{\varphi \geq t\} \quad \text{and} \quad |E| = c(m_0) \quad \text{(bathtub principle)}$$

  for a given $t > 0$.
- E is defined in a unique way since the level sets of φ have zero measure.
- The expected conclusion follows.

Bang-bang property of minimizers

Proposition (Lou-Yanagida 2006, Derlet-Gossez-Takac 2010)

Problem (P_{Dyn}) has a solution. Moreover, every minimizer m satisfies

$$\int_\Omega m = -m_0|\Omega| \qquad \text{and} \qquad m = \kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}.$$

Shape optimization version of the problem

Consequence : the two problems

$$\inf\left\{\lambda(m), \quad m \in L^\infty(\Omega,[-1,\kappa]),\ |\{m>0\}|>0,\ \int_\Omega m \leq -m_0|\Omega|\right\} \tag{1}$$

and

$$\inf\left\{\lambda(E) := \lambda(\kappa\mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}), \quad |E| = c|\Omega|\right\}, \tag{2}$$

where $c = c(m_0) \in (0,1)$, are equivalent. Moreover, each infimum is in fact a minimum.

State of the art (Highly non-exhaustive)

Proposition (Lou-Yanagida 2006, Derlet-Gossez-Takac 2010)

Problem (P_{Dyn}) has a solution. Moreover, every minimizer m satisfies

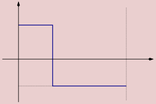
$$\int_\Omega m = -m_0|\Omega| \qquad \text{and} \qquad m = \kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}.$$

- Dirichlet case, with no sign changement on m : symmetrization, regularity in case of symmetry [Krein 1955, Friedland 1977, Cox 1990]
- Periodic case : symmetrization techniques [Hamel-Roques 2007]
- Neumann 1D case : solved [Lou-Yanagida 2006]
- Robin 1D case : optimization among intervals [Hintermüller-Kao-Laurain 2012]
- Dirichlet 2D case : regularity [Chanillo-Kenig-To 2008]
- Numerics : [Cox, Hamel-Roques, Hintermüller-Kao-Laurain]
- Asymptotics : [Mazzoleni, Pellacci, Verzini 2019] **($m_- \leq m \leq 1$ and $m_- \to -\infty$)**

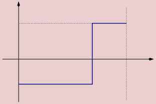
Conjectures in the Neumann case

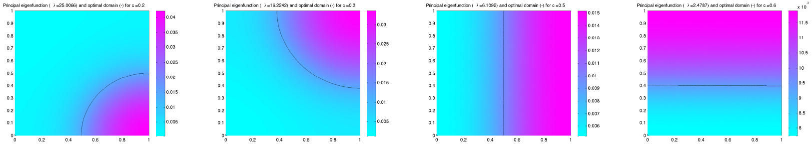
Proposition (Lou & Yanagida 2006)

In 1D (Neumann case), the only solutions of $\inf\left\{\lambda(\kappa\mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}),\ |E| = c|\Omega|\right\}$ are



and





(a) $c = 0.2$ (b) $c = 0.3$ (c) $c = 0.5$ (d) $c = 0.6$

Figure – $\Omega = (0,1)^2$. Optimal domains with $\kappa = 0.5$ and $c \in \{0.2, 0.3, 0.4, 0.5, 0.6\}$

Conjecture (Berestycki - Hamel - Roques)

For c small enough, the free boundaries of minimizers are pieces of circles.

Outline

New results : in dimension $N \geq 2$, is the solution a part of ball ?

$$\inf \left\{\lambda(E) := \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}), \quad |E| = c|\Omega|\right\} \tag{P}$$

Theorem (Lamboley, Laurain, Nadin, YP)

Let assume that $N \geq 2$ and Ω is connected and C^1. Let E is a critical point for Problem (P). Then, If E or its complement set in Ω is invariant by rotation, then Ω is a ball.

$\rightsquigarrow$The wording "critical" means that E satisfies the 1st order optimality conditions, i.e.

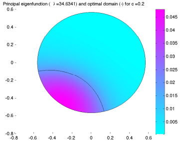
$$\text{shape derivative of } \lambda \text{ at } E \text{ in direction } V = \langle d\lambda(E), V\rangle \geq 0,$$

for all smooth vector fields $V : \mathbb{R}^N \to \mathbb{R}^N$.

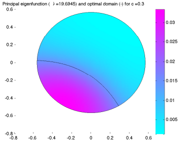
It also rewrites : E is a level set of φ, i.e. $E = \{\varphi > \alpha\}$.

Neumann case with $\Omega = B(0,1)$

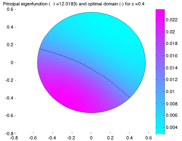
$$\inf \left\{ \lambda(E) := \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}), \quad |E| = c|\Omega| \right\} \tag{P}$$



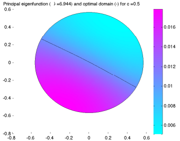
(a) $c = 0.2$



(b) $c = 0.3$



(c) $c = 0.4$



(d) $c = 0.5$

$\rightsquigarrow$ **Theorem :** the centered ball of volume $c|\Omega|$ is not a minimizer for Problem (P) (Lamboley, Laurain, Nadin, YP).

Outline

Maximizing the total population size (1)

$$\boxed{\sup_{m\in\mathcal{M}_{m_0,\kappa}} \int_\Omega u^*} \quad \text{where } u^* \text{ solves the PDE } \left\{\begin{array}{ll} \mu\Delta u^* + u^*(m-u^*) = 0 & \text{in } \Omega \\ \partial_n u^* = 0 & \text{on } \partial\Omega \end{array}\right.$$

- The existence of an optimal control m^* in $\mathcal{M}_{m_0,\kappa}$ follows from an easy compactness argument.
- Question : is any optimal control bang-bang ? (in other words, can one write $m^* = \kappa\mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}$ with E measurable ?)

Theorem (K. Nagahara and E. Yanagida, Calc. Var. PDE, 2018)

The optimal distribution m^* is such that $\{-1 < m^* < \kappa\}$ does not contain any open set.

$\rightsquigarrow$ maximizers are bang-bang under a "regularity" assumption on m^*.

Maximizing the total population size (1)

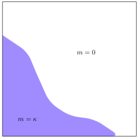
$$\boxed{\sup_{m\in\mathcal{M}_{m_0,\kappa}} \int_\Omega u^*}$$ where u^* solves the PDE $\begin{cases} \mu\Delta u^* + u^*(m-u^*) = 0 & \text{in } \Omega \\ \partial_n u^* = 0 & \text{on } \partial\Omega \end{cases}$

Theorem (Mazari, Nadin, YP)

Let Ω be a bounded connected domain with $\mathcal{C}^2$ boundary.

- Every solution of the problem above writes $m^* = \mathbb{1}_{E_\mu}$.
- If μ is small enough, optimal domains are "fragmented".
- In 1D, if $\mu \geq \mu^*$: E_μ is an interval meeting one extremity of Ω

$\rightsquigarrow$ Similar conclusions for general domains Ω



Maximizing the total population size (2)

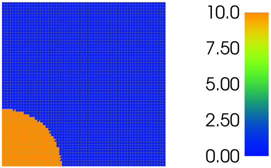
$$\boxed{\sup_{|E|=c|\Omega|} \int_\Omega u^*}$$ where u^* solves the PDE $\begin{cases} \mu\Delta u^* + u^*(\kappa \mathbb{1}_E - u^*) = 0 & \text{in } \Omega \\ \partial_n u^* = 0 & \text{on } \partial\Omega \end{cases}$

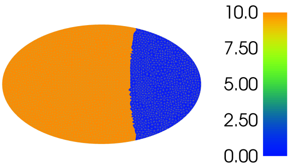
$\rightsquigarrow$ In this model, we always have persistence of species (i.e. $u(t,\cdot) \to u^*$ as $t \to +\infty$)

Theorem (Mazari, Nadin, YP)

Let Ω be a convex domain. As $\mu \to +\infty$, E_μ converges in the sense of characteristic functions to a solution of the shape optimization problem

$$\sup_{|E|=c|\Omega|} \int_\Omega |\nabla u^\infty|^2 \quad \text{where } u^\infty \text{ solves the PDE } \begin{cases} \Delta u^\infty + c(\kappa \mathbb{1}_E - c) = 0 & \text{in } \Omega \\ \int_\Omega u^\infty = 0, \quad \partial_n u^\infty = 0 & \text{on } \partial\Omega \end{cases}$$





Sketch of proof : existence of optimal shapes

Let u^* be the solution of $\begin{cases} \mu\Delta u^* + u^*(m - u^*) = 0 & \text{in } \Omega \\ \partial_n u^* = 0 & \text{on } \partial\Omega \end{cases}$

- Computation of the second order derivative : we set $F_\mu(m) = \int_\Omega u$.

$$\ddot{F}_\mu(m)[h,h] = \int_\Omega \ddot{u} = \int_\Omega V_1 |\nabla \dot{u}|^2 - \int_\Omega V_2 \dot{u}^2$$

with $V_1(\cdot)$ positively bounded by below, V_2 in $L^\infty(\Omega)$, where

$$\begin{cases} \mu\Delta \dot{u} + (m - 2u^*)\dot{u} = -hu^* & \text{in } \Omega, \\ \frac{\partial \dot{u}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence,

$$\exists A_1,\ A_2 > 0 \quad | \quad \ddot{F}_\mu(m)[h,h] \geq A_1 \int_\Omega |\nabla \dot{u}|^2 - A_2 \int_\Omega \dot{u}^2$$

Sketch of proof : existence of optimal shapes

Let u^* be the solution of $\begin{cases} \mu\Delta u^* + u^*(m - u^*) = 0 & \text{in } \Omega \\ \partial_n u^* = 0 & \text{on } \partial\Omega \end{cases}$

- Our goal is now to construct an admissible perturbation $h \in L^\infty(\Omega)$ such that

$$h \text{ is supported in } \{0 < m < \kappa\}, \qquad \ddot{F}_\mu(m)[h, h] > 0.$$

- In that case a Taylor expansion yields

$$F_\mu(m + \varepsilon h) - F_\mu(m) = \frac{\varepsilon^2}{2}\ddot{F}_\mu(m)[h, h] + o(\varepsilon^2)$$

  which leads to a contradiction whenever $\varepsilon > 0$ is chosen small enough

- To obtain a contradiction, it hence suffices to construct a perturbation $h \in L^2(\Omega)$ with support in $\{0 < m < \kappa\}$ satisfying $\int_\Omega h = 0$ and such that

$$\int_\Omega |\nabla \dot{u}|^2 > \frac{A_2}{A_1}\int_\Omega \dot{u}^2.$$

Sketch of proof : existence of optimal shapes

Let u^* be the solution of $\begin{cases} \mu\Delta u^* + u^*(m-u^*) = 0 & \text{in } \Omega \\ \partial_n u^* = 0 & \text{on } \partial\Omega \end{cases}$

- We expand $-hu$ as the high frequencies series

$$-hu = \sum_{\ell \geq K+1} \alpha_\ell \psi_\ell \quad \text{with} \quad \begin{cases} -\mu\Delta\psi_k - (m-2u^*)\psi_k = \lambda_k\psi_k \text{ in } \Omega, \\ \frac{\partial \psi_k}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ \int_\Omega \psi_k^2 = 1. \end{cases}$$

- Then,

$$\dot{u} = \sum_{\ell \geq K+1} \frac{\alpha_\ell}{\lambda_\ell}\psi_\ell.$$

and we infer the existence of $M > 0$ such that

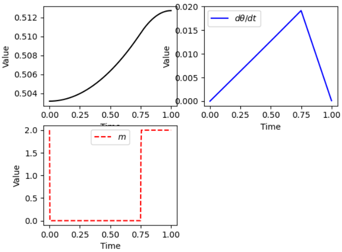
$$\int_\Omega |\nabla \dot{u}|^2 \geq (\lambda_{K+1} - M)\int_\Omega \dot{u}^2$$

and we are done.

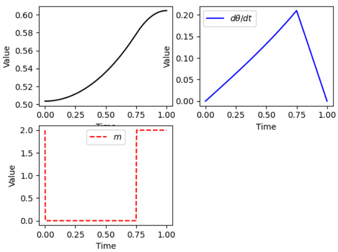
Numerics : optimal control in the 1D case

θ : solution of the steady-state problem ($\mu\Delta u + u(m - u) = 0$ in $+$ Neumann B.C.)

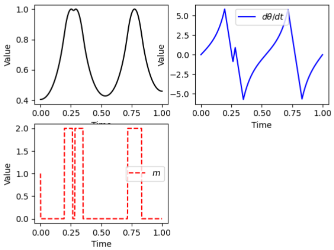
Paramètres : μ =10, $m_0 =$ 0.5, $\mu =$ 10



Paramètres : μ =1, $m_0 =$ 0.5, $\mu =$ 1

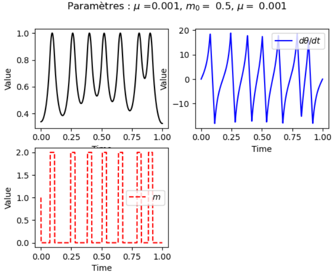


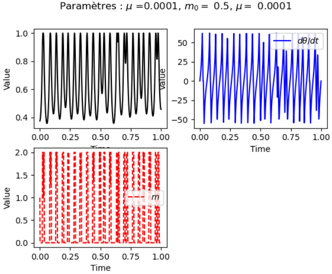
Paramètres : μ =0.01, $m_0 =$ 0.5, $\mu =$ 0.01



Numerics : optimal control in the 1D case

θ : solution of the steady-state problem ($\mu\Delta u + u(m-u) = 0$ in + Neumann B.C.)





Theorem (Mazari, Nadin, YP)

Let $d \geq 1$ and let $= (0;1)^d$. There exists $C_0 > 0$ such that the following holds : there exists $\mu_0 > 0$ such that, for any $\mu \in (0, \mu_0)$, then

$$\|m\|_{BV(\Omega)} \geq \frac{C_0}{\sqrt{\mu}}.$$

Outline

Similar problem when adding a drift term

$\rightsquigarrow$ We enrich the model by

- adding an advection term along the gradient of the habitat quality (according to Belgacem and Cosner)
- considering Robin type boundary conditions

$$\begin{cases} \partial_t u = \operatorname{div}(\nabla u - \alpha u \nabla m) + \lambda u(m-u) & \text{in } \Omega \times (0,\infty), \\ e^{\alpha m}(\partial_n u - \alpha u \partial_n m) + \beta u = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

This models the tendency of the population to move up along the gradient of m.

New shape optimization problem

$$\inf_{m \in \mathcal{M}_{m_0,\kappa}} \lambda_\alpha(m),$$

$$\text{with} \quad \lambda_\alpha(m) = \inf_{\varphi \in \mathcal{S}_0} \frac{\int_\Omega e^{\alpha m} |\nabla \varphi|^2 + \beta \int_{\partial\Omega} \varphi^2}{\int_\Omega m e^{\alpha m} \varphi^2}$$

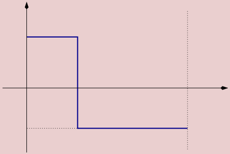
and $\mathcal{S}_0 = \{\varphi \in H^1(\Omega),\ \int_\Omega m e^{\alpha m} \varphi^2 > 0\}$.

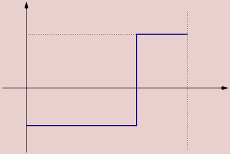
Similar problem when adding a drift term

Theorem (1D model, Caubet, Deheuvels, YP (2017))

Assume that $\Omega = (0,1)$. There exists $\beta^* > 0$ such that

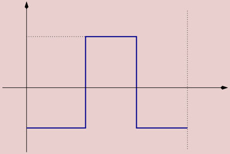
- **if** $\beta < \beta^*$,





are the only solutions.

- **if** $\beta > \beta^*$



is the only solution.

F. Caubet, T. Deheuvels, Y. Privat, *Optimal location of resources for biased movement of species : the 1D case*, SIAM J. Applied Math 77 (2017), no. 6, 1876–1903.

Similar problem when adding a drift term

Theorem (Mazari, Nadin, YP (2019))

Assume that $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ is bounded and connected.

- If the problem
$$\inf_{m \in \mathcal{M}_{m_0,\kappa}} \lambda_\alpha(m)$$
has a solution m^*, then necessarily, m^* is bang-bang (i.e. $\exists E^* \subset \Omega$ s.t. $m^* = \kappa \mathbb{1}_{E^*}$)
- In that case, if moreover ∂E^* is a C^2 hypersurface, then Ω is necessarily a ball.
- If Ω is a ball, if α is small enough and if $n = 2, 3$, the centered ball is the unique minimizer of $E \mapsto \lambda_\alpha(\mathbb{1}_E)$ among radially symmetric domains E with prescribed volume $c|\Omega|$.

Open problem : case where Ω is a ball.

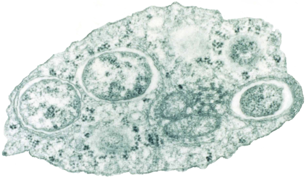
Existence and characterization of optimal radially symmetric domains in any dimension ?

I. Mazari, G. Nadin, Y. Privat, *Shape optimization of a two-phase weighted Dirichlet eigenvalue*, To apear in ARMA.

Outline

Optimal releases for population replacement strategies

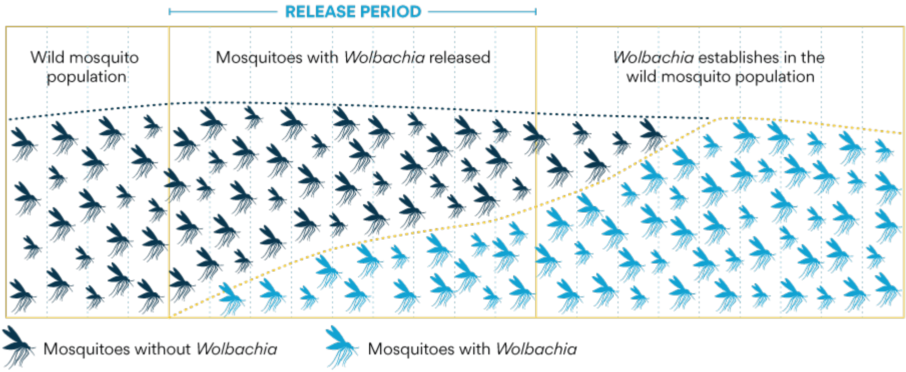
- Endo-symbiotic bacteria ;
- Reproduction manipulators ;
- **Cytoplasmic incompatibility** (CI) ;
- **Pathogen interference**= vector competence suppression for key pathogens (dengue, zika, chikungunya viruses) in *Ae. spp.* ;
- Typically reduces fecundity and life-span.



♀\♂	Infected	Sound
Infected	I	I
Sound	$\times$	S

Wolbachia replacement concept

Development : From a research project (Scott O'Neill at Monash University (Australia), first project in 2011) to the not-for-profit initiative Eliminate Dengue "World Mosquito Project" with projects in 12 countries : India, Sri Lanka, Vietnam, Indonesia, Australia, Kiribati, New Caledonia, Vanuatu, Fiji, Mexico, Colombia, Brazil.
Target : *Aedes aegypti* (main dengue vector).
(Source : `http://www.eliminatedengue.com/program`)



Model motivation

The first model dates back to Caspari and Watson (1959), considering (in an infinite population) only the **frequency of infected individuals** p. It shows the bistable nature of the system.

Problem

When dealing with control, what is appropriate ? We release infected *individuals* so the induced variations on p are non-trivial.

Model

With constant sex-ratio and further simplification of life-cycle, $\mathrm{n} = (n_1, n_2) \in \mathbb{R}_+^2$ (wild and infected) :

$$\begin{cases} \dfrac{dn_1}{dt} - \nu\Delta n_1 = f_1(n_1, n_2) := b_1 n_1 \left(1 - s_h \dfrac{n_2}{n_1 + n_2}\right)\left(1 - \dfrac{n_1 + n_2}{K}\right) - d_1 n_1, \\ \dfrac{dn_2}{dt} - \nu\Delta n_2 = f_2(n_1, n_2) := b_2 n_2 \left(1 - \dfrac{n_1 + n_2}{K}\right) - d_2 n_2 + u, \\ \text{B.C.} + (n_1(0), n_2(0)) = (X_1, 0). \end{cases} \tag{3}$$

where $s_h \in (0, 1]$ is CI rate, K is environmental capacity, b_i and d_i are birth and death rates.

In compact form : $\boxed{\dot{\mathrm{n}}_u - \nu\Delta u = \mathrm{f}(\mathrm{n}_u) + u \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$.

Principle : Two competing populations (for breeding sites), with reproductive interference by (unidirectional) CI are exposed to releases $u \geq 0$.

Admissible controls, reachable set

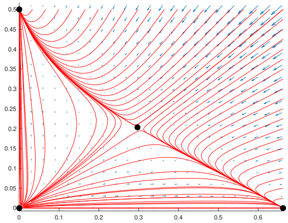
Controls $u : [0, T] \to [0, +\infty)$ are taken in

$$\boxed{\mathcal{U}_{T,C,M} := \left\{ u : [0,T] \to \mathbb{R} \text{ measurable}, \ 0 \leq u \leq M, \ \int_0^T u(t)dt \leq C \right\}}.$$

$\mathcal{U}_{T,C,M}$ is compact for L^∞-weak* topology. (Banach-Alaoglu)

Interpretation : controls are bounded in

- L^1 : total number of individuals to release is less than C ;
- L^∞ : release rate (individuals per unit of time) is less than M.



Phase portrait of the associated ODE

⚠ Difficulties of determining persistence/extinction criteria

$\boxed{(0, n_2^*)}$: invasion state (locally linearly stable)

We want to minimize, for u in $\mathcal{U}_{T,C,M}$:

$$\boxed{J(u) = \frac{1}{2} n_1(T)^2 + \frac{1}{2} \left(n_2^* - n_2(T)\right)_+^2}$$

Thank you for your attention