Shape optimization problem involving the Tresca friction law in a scalar case

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Some references

- G. Allaire, F. Jouve, and A. Maury. **Shape optimisation with the level set method for contact problems in linearised elasticity.** The SMAI journal of computational mathematics, 3:249–292, 2017
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Summary

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Summary

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- The optimization variables are subset of $\mathbb{R}^d$.
- In order to numerically minimizes the cost functional, we need to compute its **shape gradient**.

Summary

Physical context

- **Mechanical contact** describes the contact of deformable solids that touch each other on parts of their boundaries without penetration and possibly sliding with friction.
- Mathematical model: elastic body, Signorini's law, Tresca's friction law.

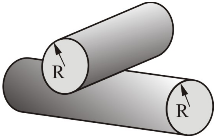


Figure: Contact between cylinders with radius $R > 0$.

Model presentation

Let Ω be a nonempty bounded connected open subset of $\mathbb{R}^d$, $d \in \mathbb{N}^*$.
The stress vector of Ω is defined by

$$T(u) := \sigma_{\mathrm{n}}(u)\mathrm{n} + \sigma_\tau(u),$$

where $u : \Omega \to \mathbb{R}^d$ is the displacement field and:

- $\sigma_{\mathrm{n}}(u)$ is the normal stress;
- $\sigma_\tau(u)$ is the shear stress.

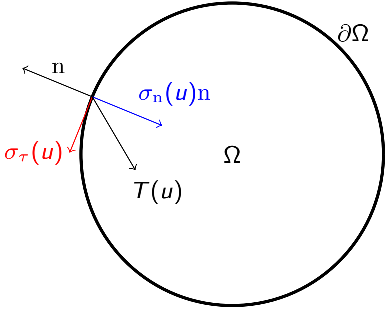


Figure: An elastic body Ω for $d = 2$.

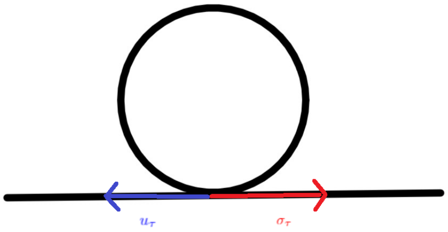
Tresca friction law

Definition (Tresca friction law)

Let Ω defined as above and $\partial\Omega_T \subset \partial\Omega$. The Tresca friction law defined on $\partial\Omega_T$:

$$\left\{\begin{array}{rcll} \|\sigma_\tau\| & \leq & g, & \\ \text{if } \|\sigma_\tau\| & < & g, & \text{then } u_\tau = 0; \\ \text{if } \|\sigma_\tau\| & = & g, & \text{then it exists } \lambda \geq 0 \text{ such as } u_\tau = -\lambda\sigma_\tau, \end{array}\right.$$

where g is a positive function on $\partial\Omega_T$ called the friction threshold.



- Equivalent formulation of the Tresca friction law:

$$\|\sigma_\tau\| \leq g \text{ et } u_\tau \cdot \sigma_\tau = -g\,\|u_\tau\|\,.$$

- If u is a scalar valued function then

$$|\partial_{\mathrm{n}} u| \leq g \text{ and } u\partial_{\mathrm{n}} u = -g\,|u|\,.$$

Summary

Objective

Let $d \in \mathbb{N}^*$, $f \in \mathrm{H}^1(\mathbb{R}^d)$ and $g \in \mathrm{H}^2(\mathbb{R}^d)$ such as $g > 0$ a.e. on $\mathbb{R}^d$.
We consider the shape optimization problem:

$$\min_{\substack{\Omega \in \mathcal{U}_{ad} \\ |\Omega| = \lambda}} \mathcal{J}(\Omega),$$

where

$$\mathcal{U}_{ad} := \Big\{ \Omega \subset \mathbb{R}^d \mid \Omega \text{ nonempty connected bounded open subset of } \mathbb{R}^d \text{ with } \mathcal{C}^3\text{-boundary} \Big\},$$

with the volume constraint $|\Omega| = \lambda > 0$ and

$\mathcal{J} : \mathcal{U}_{ad} \to \mathbb{R}$ is the *energy functional* defined by

$$\mathcal{J}(\Omega) := \frac{1}{2} \int_{\Omega} \left(\|\nabla u_\Omega\|^2 + |u_\Omega|^2 \right) + \int_{\partial\Omega} g|u_\Omega| - \int_{\Omega} f u_\Omega,$$

and $u_\Omega \in \mathrm{H}^1(\Omega)$ stands for the unique solution to the Tresca friction problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ |\partial_{\mathrm{n}} u| \leq g \text{ and } u\partial_{\mathrm{n}} u + g|u| = 0 & \text{on } \partial\Omega. \end{cases} \qquad (\mathrm{TP}_\Omega)$$

Tresca friction problem

Definition (Weak solution to the Tresca friction problem)

A weak solution to the Tresca friction problem is a function $u : \Omega_0 \to \mathbb{R}$ such that $u \in \mathrm{H}^1(\Omega_0)$ and for all $v \in \mathrm{H}^1(\Omega_0)$,

$$\int_{\Omega_0} \nabla u \cdot \nabla(v-u) + \int_{\Omega_0} u(v-u) + \phi(v) - \phi(u) \geq \int_{\Omega_0} f(v-u).$$

Definition (Tresca functional)

The Tresca friction functional ϕ is defined

$$\begin{array}{rccl} \phi : & \mathrm{H}^1(\Omega_0) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \displaystyle\int_{\partial\Omega_0} g|v|. \end{array}$$

Tresca friction problem

Proposition (Existence and uniqueness)

The Tresca friction problem admits a unique solution characterized by

$$u = \mathrm{prox}_{\phi}(F),$$

where F is the solution the unique solution to the Neumann problem

$$\begin{cases} -\Delta F + F = f & \text{in } \Omega_0, \\ \partial_{\mathrm{n}} F = 0 & \text{on } \partial\Omega_0. \end{cases}$$

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Proof.

Note that $\phi \in \Gamma(\mathrm{H}^1(\Omega_0))$. Using the weak formulation of the Neumann problem, u is a weak solution to the Tresca friction problem iff,

$$\phi(\varphi) \geq \langle F - u, \varphi - u \rangle_{\mathrm{H}^1(\Omega_0)} + \phi(u), \qquad \forall \varphi \in \mathrm{H}^1(\Omega_0)$$

i.e iff $F - u \in \partial\phi(u)$,
i.e iff $u = \operatorname{prox}_\phi(F)$. $\square$

Summary

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Shape gradient

Let $\Omega_0 \in \mathcal{U}_{ad}$ be an initial shape, if $\theta \in \mathcal{C}^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ then $\mathbf{id} + \boldsymbol{\theta}$ is $\mathcal{C}^3$-diffeomorphism and $\Omega_\theta := (\mathbf{id} + \boldsymbol{\theta})(\Omega_0) \in \mathcal{U}_{ad}$.

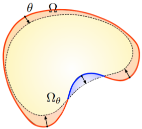

Figure: The shape perturbated Ω_θ.[1]

$\mathcal{J}$ is shape differentiable at Ω_0 if

$$\begin{array}{rccl} \xi : & \mathcal{C}^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & \boldsymbol{\theta} & \longmapsto & \xi(\boldsymbol{\theta}) := \mathcal{J}((\mathbf{id} + \boldsymbol{\theta})(\Omega_0)), \end{array}$$

is Gateaux differential at 0, and we denote by $\mathcal{J}'(\Omega_0) := d_G\xi(0)$ the shape gradient of $\mathcal{J}$ at Ω_0.

[1] C. Dapogny, *An introduction to shape and topology optimization.*

- $\Omega_0 \in \mathcal{U}_{ad}$, $\boldsymbol{\theta} \in \mathcal{C}^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ and $t > 0$ sufficiently small such that $\Omega_t := (\mathbf{id} + t\boldsymbol{\theta})(\Omega_0) \in \mathcal{U}_{ad}$.

We denote $u_t \in \mathrm{H}^1(\Omega_t)$ the shape perturbed Tresca friction solution on Ω_t:

$$\begin{cases} -\Delta u_t + u_t = f & \text{in } \Omega_t, \\ |\partial_{\mathrm{n}} u_t| \leq g \text{ and } u_t \partial_{\mathrm{n}} u_t + g|u_t| = 0 & \text{on } \partial\Omega_t, \end{cases}$$

with the weak formulation

$$\int_{\Omega_t} \nabla u_t \cdot \nabla(v - u_t) + \int_{\Omega_t} u_t(v - u_t) + \int_{\partial\Omega_t} g|v| - \int_{\partial\Omega_t} g|u_t| \geq \int_{\Omega_t} f(v - u_t), \qquad \forall v \in \mathrm{H}^1(\Omega_t).$$

Differentiability of $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega_t)$?

Change of variables $\mathbf{id} + t\boldsymbol{\theta}$, $\overline{u}_t := u_t \circ (\mathbf{id} + t\boldsymbol{\theta}) \in \mathrm{H}^1(\Omega_0)$ is solution to:

$$\int_{\Omega_0} \mathrm{A}_t \nabla \overline{u}_t \cdot \nabla (v - \overline{u}_t) + \int_{\Omega_0} \overline{u}_t (v - \overline{u}_t) \mathrm{J}_t + \int_{\partial\Omega_0} g_t \mathrm{J}_{\mathrm{T}_t} |v| - \int_{\partial\Omega_0} g_t \mathrm{J}_{\mathrm{T}_t} |\overline{u}_t| \geq \int_{\Omega_0} f_t \mathrm{J}_t (v - \overline{u}_t), \qquad \forall v \in \mathrm{H}^1(\Omega_0),$$

where

- $f_t := f \circ (\mathbf{id} + t\boldsymbol{\theta}) \in \mathrm{H}^1(\mathbb{R}^d)$,
- $g_t := g \circ (\mathbf{id} + t\boldsymbol{\theta}) \in \mathrm{H}^2(\mathbb{R}^d)$,
- $\mathrm{J}_t := \det(\mathrm{I} + t\nabla\boldsymbol{\theta}) \in \mathrm{L}^\infty(\mathbb{R}^d)$,
- $\mathrm{A}_t := \det(\mathrm{I} + t\nabla\boldsymbol{\theta})(\mathrm{I} + t\nabla\boldsymbol{\theta})^{-1}(\mathrm{I} + t\nabla\boldsymbol{\theta}^\top)^{-1} \in \mathrm{L}^\infty(\mathbb{R}^d, \mathbb{R}^{d\times d})$,
- $\mathrm{J}_{\mathrm{T}_t} := \det(\mathrm{I} + t\nabla\boldsymbol{\theta}) \|(\mathrm{I} + t\nabla\boldsymbol{\theta}^\top)^{-1}\mathrm{n}\| \in \mathcal{C}^0(\partial\Omega_0)$.

Thus,

$$\overline{u}_t = \mathrm{prox}_{\phi(t,\cdot)}(F_t),$$

where $F_t \in \mathrm{H}^1(\Omega_0)$ stands for the unique solution to the perturbed Neumann problem

$$\langle F_t, v\rangle_{\mathrm{H}^1(\Omega_0)} = \int_{\Omega_0} f_t \mathrm{J}_t v - \int_{\Omega_0} (\mathrm{A}_t - \mathrm{I}) \nabla \overline{u}_t \cdot \nabla v - \int_{\Omega_0} (\mathrm{J}_t - 1)\, \overline{u}_t v, \qquad \forall v \in \mathrm{H}^1(\Omega_0),$$

and $\mathrm{prox}_{\phi(t,\cdot)} : (\mathrm{H}^1(\Omega_0), \langle\cdot,\cdot\rangle_{\mathrm{H}^1(\Omega_0)}) \to (\mathrm{H}^1(\Omega_0), \langle\cdot,\cdot\rangle_{\mathrm{H}^1(\Omega_0)})$ is the proximal operator associated with the perturbed Tresca friction functional

$$\begin{array}{rccl} \Phi : & \mathbb{R}_+ \times \mathrm{H}^1(\Omega_0) & \longrightarrow & \mathbb{R} \\ & (t, v) & \longmapsto & \Phi(t, v) := \displaystyle\int_{\partial\Omega_0} g_t \mathrm{J}_{\mathrm{T}_t} |v|. \end{array}$$

Lemma

The map $t \in \mathbb{R}_+ \mapsto F_t \in \mathrm{H}^1(\Omega_0)$ is differentiable at $t = 0$, and its derivative $F_0' \in \mathrm{H}^1(\Omega_0)$ is the unique solution to the Neumann problem

$$\left\langle F_0', v\right\rangle_{\mathrm{H}^1(\Omega_0)} = \int_{\Omega_0} \left(f\mathrm{div}(\boldsymbol{\theta}) + \nabla f \cdot \boldsymbol{\theta}\right) v$$

$$-\int_{\Omega_0} \left(-\nabla\boldsymbol{\theta} - \nabla\boldsymbol{\theta}^\top + \mathrm{div}(\boldsymbol{\theta})\mathrm{I}\right) \nabla u_0 \cdot \nabla v - \int_{\Omega_0} \mathrm{div}(\boldsymbol{\theta}) u_0 v, \qquad \forall v \in \mathrm{H}^1(\Omega_0).$$

Proof.

- $t \in \mathbb{R}_+ \mapsto \mathrm{J}_t \in \mathrm{L}^\infty(\mathbb{R}^d)$ is differentiable at $t = 0$ with its derivative $\mathrm{div}(\boldsymbol{\theta})$;
- $t \in \mathbb{R}_+ \mapsto f_t \mathrm{J}_t \in \mathrm{L}^2(\mathbb{R}^d)$ is differentiable at $t = 0$ with its derivative $f\mathrm{div}(\boldsymbol{\theta}) + \nabla f \cdot \boldsymbol{\theta}$;
- $t \in \mathbb{R}_+ \mapsto \mathrm{A}_t \in \mathrm{L}^\infty(\mathbb{R}^d, \mathbb{R}^{d\times d})$ is differentiable at $t = 0$ with its derivative $\mathrm{A}_0' := -\nabla\boldsymbol{\theta} - \nabla\boldsymbol{\theta}^\top + \mathrm{div}(\boldsymbol{\theta})\mathrm{I}$;
- $t \in \mathbb{R}_+ \mapsto g_t \mathrm{J}_{\mathrm{T}_t} \in \mathrm{L}^2(\partial\Omega_0)$ is differentiable at $t = 0$ with its derivative $\nabla g \cdot \boldsymbol{\theta} + g(\mathrm{div}(\boldsymbol{\theta}) - \nabla\boldsymbol{\theta}\mathbf{n} \cdot \mathbf{n})$.

□

Differentiability of $t \in \mathbb{R}_+ \mapsto \overline{u}_t = \mathrm{prox}_{\Phi(t,\cdot)}(F_t) \in \mathrm{H}^1(\Omega_0)$ at $t = 0$.

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Theorem (S. Adly, L. Bourdin (2018))

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be a Hilbert space and $\Phi : \mathbb{R}^+ \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ such that for all $t \geq 0$, $\Phi(t, \cdot) \in \Gamma(\mathcal{H})$. Let $y : \mathbb{R}^+ \to \mathcal{H}$ and $x : \mathbb{R}^+ \to \mathcal{H}$ be defined by

$$x(t) := \mathrm{prox}_{\Phi(t,\cdot)}(y(t)),$$

for all $t \geq 0$. If the following conditions are satisfied:

1. *y is differentiable at $t = 0$;*
2. *Φ is twice epi-differentiable at $x(0)$ for $y(0) - x(0) \in \partial\Phi(0, \cdot)(x(0))$;*
3. *$d_e^2\phi(x(0)|y(0) - x(0))$ is proper function.*

Then x is differentiable at $t = 0$ with

$$x'(0) = \mathrm{prox}_{d_e^2\Phi(x(0)|y(0)-x(0))}(y'(0)).$$

Mosco epi-convergence

Proposition (Sequential characterization)

Let us consider $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ *a Hilbert space,* $(f_t)_{t\geq 0}$ *a parameterized family of functions* $f_t : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$*, for all* $t \geq 0$*, and* $f : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$.
Then $(f_t)_{t\geq 0}$ *Mosco epi-converges to* f*, if for all* $x \in \mathcal{H}$ *the two conditions*

1. *it exists* $(x_t)_{t\geq 0} \to x$ *such that* $\limsup f_t(x_t) \leq f(x)$*;*
2. *for all* $(x_t)_{t\geq 0} \rightharpoonup x$, $\liminf f_t(x_t) \geq f(x)$*;*

are satisfied.
In that case we denote $\mathrm{ME\text{-}lim}\ f_t := f$ *the Mosco epi-limite of the parameterized family* $(f_t)_{t\geq 0}$.

Twice epi-differentiability

Definition (S. Adly, L. Bourdin (2018))

Let $f : \mathbb{R}^+ \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ *such that for all* $t \geq 0$, $f(t,\cdot) \in \Gamma(\mathcal{H})$. f *is said to be twice epi-differentiable at* $x \in f^{-1}(\cdot,\mathbb{R})$ *for* $y \in \partial f(0,\cdot)(x)$ *if* $(\Delta_t f(x|y))_{t>0}$ *defined by*

$$\begin{array}{rccl} \Delta_t f(x|y) : & \mathcal{H} & \longrightarrow & \mathbb{R} \cup \{+\infty\} \\ & z & \longmapsto & \dfrac{f(t,x+tz) - f(t,x) - t\left\langle y,z\right\rangle_{\mathcal{H}}}{\frac{1}{2}t^2}, \end{array}$$

for all $t > 0$, *is Mosco epi-convergent. In that case we denote*

$$\mathrm{d}_e^2 f(x|y) := \mathrm{ME\text{-}lim}\ \Delta_t f(x|y),$$

which is called the second-order epi-derivative of f *at* x *for* y.

If f is t-independent and twice differentiable at x then

$$\mathrm{d}_e^2 f(x|\nabla f(x))(z) = \mathrm{D}^2 f(x)(z,z), \qquad \forall z \in \mathcal{H}.$$

Reminder:

$$\begin{array}{rccl} \Phi : & \mathbb{R}_+ \times \mathrm{H}^1(\Omega_0) & \longrightarrow & \mathbb{R} \\ & (t,v) & \longmapsto & \Phi(t,v) := \displaystyle\int_{\partial\Omega_0} g_t \mathrm{J}_{\mathrm{T}_t} |v|, \end{array}$$

Proposition (Second-order difference quotient functions of Φ)

For all $t > 0$, $u \in \mathrm{H}^1(\Omega_0)$, $v \in \partial\Phi(0,\cdot)(u)$ one has

$$\Delta_t \Phi(u|v)(w) = \int_{\partial\Omega_0} \Delta_t G(s)(u(s)|\partial_{\mathrm{n}} v(s))(w(s))\mathrm{d}s, \qquad \forall w \in \mathrm{H}^1(\Omega_0),$$

where for almost all $s \in \partial\Omega_0$, $G(s)$ is defined by

$$\begin{array}{rccl} G(s) : & \mathbb{R}_+ \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & (t,x) & \longmapsto & G(s)(t,x) := g_t(s)\mathrm{J}_{\mathrm{T}_t}(s)|x|. \end{array}$$

Proposition (Twice epi-differentiability of G)

For almost all $s \in \partial\Omega_0$, if g has a directional derivative at s in any direction. Then, for all $x \in \mathbb{R}$ and $y \in \partial G(s)(0,\cdot)(x) = g(s)\partial|\cdot|(x)$,

$$\mathrm{d}_e^2 G(s)(x|y)(z) = \iota_{\mathrm{K}_{x,\frac{y}{g(s)}}}(z) + \left(\nabla g(s)\cdot\boldsymbol{\theta}(s) + g(s)\left(\mathrm{div}(\boldsymbol{\theta}(s)) - \nabla\boldsymbol{\theta}(s)\mathbf{n}(s)\cdot\mathbf{n}(s)\right)\right)\frac{y}{g(s)}z, \quad (1)$$

for all $z \in \mathbb{R}$, where $\mathrm{d}_e^2 G(s)(x|y)$ is the second-order epi-derivative of $G(s)$ at x for y et où

$$\mathrm{K}_{x,y} := \begin{cases} \mathbb{R} & \text{si } x \neq 0, \\ \mathbb{R}^+ & \text{si } x = 0 \text{ et } y = 1, \\ \mathbb{R}^- & \text{si } x = 0 \text{ et } y = -1, \\ \{0\} & \text{si } x = 0 \text{ et } y \in (-1,1). \end{cases}$$

Proof.

$$\Delta_t G(s)(x|y)(z) = g_t(s)\mathrm{J}_{\mathrm{T}_t}(s)\frac{|x+tz|-|x|-t\frac{y}{g(s)}z}{t^2} + \frac{g_t(s)\mathrm{J}_{\mathrm{T}_t}(s)-g(s)}{t}\frac{y}{g(s)}z.$$

□

Theorem (S. Adly, L. Bourdin (2018))

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be a Hilbert space and $\Phi : \mathbb{R}^+ \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ such that for all $t \geq 0$, $\Phi(t,\cdot) \in \Gamma(\mathcal{H})$. Let $y : \mathbb{R}^+ \to \mathcal{H}$ and $x : \mathbb{R}^+ \to \mathcal{H}$ be defined by

$$x(t) := \mathrm{prox}_{\Phi(t,\cdot)}(y(t)),$$

for all $t \geq 0$. If the following conditions are satisfied :

1. *y is differentiable at $t = 0$;*
2. *Φ is twice epi-differentiable at $x(0)$ for $y(0) - x(0) \in \partial\Phi(0,\cdot)(x(0))$;*
3. *$d_e^2\phi(x(0)|y(0) - x(0))$ is proper function.*

Then x is differentiable at $t = 0$ with

$$x'(0) = \mathrm{prox}_{d_e^2\Phi(x(0)|y(0)-x(0))}(y'(0)).$$

Material derivative

Let us assume that :

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$$\mathrm{d}_e^2\Phi(u_0|F_0 - u_0)(w) = \int_{\partial\Omega_0} \mathrm{d}_e^2 G(s)(u_0(s)|\partial_{\mathrm{n}}(F_0 - u_0)(s))(w(s))\,\mathrm{d}s,$$

for all $w \in \mathrm{H}^1(\Omega_0)$.

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for all $w \in \mathrm{H}^1(\Omega_0)$.

Then the map $t \in \mathbb{R}_+ \mapsto \overline{u}_t \in \mathrm{H}^1(\Omega_0)$ is differentiable at $t = 0$ and its derivative $\overline{u}_0' \in \mathrm{H}^1(\Omega_0)$ is given by

$$\overline{u}_0' = \mathrm{prox}_{\mathrm{d}_e^2\Phi(u_0|F_0-u_0)}(F_0').$$

$$\left\langle \overline{u}_0', v - \overline{u}_0' \right\rangle_{\mathrm{H}^1(\Omega_0)} \geq \int_{\Omega_0} -\Delta\left(\boldsymbol{\theta}\cdot\nabla u_0\right)\left(v - \overline{u}_0'\right) + \int_{\Omega_0} \boldsymbol{\theta}\cdot\nabla u_0 \left(v - \overline{u}_0'\right) + \int_{\partial\Omega_0} h^m(\boldsymbol{\theta})\left(v - \overline{u}_0'\right),$$

for all $v \in \mathcal{C}_{u_0, \frac{\partial_{\mathrm{n}}(F_0 - u_0)}{g}} :=$
$$\left\{ w \in \mathrm{H}^1(\Omega_0) \mid w(s) \leq 0 \text{ on } \partial\Omega_{S-}^{u_0,g}, w(s) \geq 0 \text{ on } \partial\Omega_{S+}^{u_0,g}, w(s) = 0 \text{ on } \partial\Omega_{\mathrm{D}}^{u_0,g} \right\},$$

with

$$h^m(\boldsymbol{\theta}) := \left(\nabla g\cdot\boldsymbol{\theta} - g\nabla\boldsymbol{\theta}\mathbf{n}\cdot\mathbf{n}\right)\frac{\partial_{\mathrm{n}} u_0}{g} + (\nabla\boldsymbol{\theta} + \nabla\boldsymbol{\theta}^\top)\nabla u_0\cdot\mathbf{n} \in \mathrm{L}^2(\partial\Omega_0)$$

Material derivative

$$\begin{cases} -\Delta \overline{u}_0' + \overline{u}_0' = -\Delta\left(\boldsymbol{\theta}\cdot\nabla u_0\right) + \boldsymbol{\theta}\cdot\nabla u_0 & \text{in } \Omega_0, \\ \overline{u}_0' = 0 & \text{on } \partial\Omega_{\mathrm{D}}^{u_0,g}, \\ \partial_{\mathrm{n}}\overline{u}_0' = h^m(\boldsymbol{\theta}) & \text{on } \partial\Omega_{\mathrm{N}}^{u_0,g}, \\ \overline{u}_0' \leq 0,\ \partial_{\mathrm{n}}\overline{u}_0' \leq h^m(\boldsymbol{\theta}),\ \overline{u}_0'\left(\partial_{\mathrm{n}}\overline{u}_0' - h^m(\boldsymbol{\theta})\right) = 0 & \text{on } \partial\Omega_{\mathrm{S-}}^{u_0,g}, \\ \overline{u}_0' \geq 0,\ \partial_{\mathrm{n}}\overline{u}_0' \geq h^m(\boldsymbol{\theta}),\ \overline{u}_0'\left(\partial_{\mathrm{n}}\overline{u}_0' - h^m(\boldsymbol{\theta})\right) = 0 & \text{on } \partial\Omega_{\mathrm{S+}}^{u_0,g}, \end{cases}$$

where:

- $h^m(\boldsymbol{\theta}) := (\nabla g\cdot\boldsymbol{\theta} g\nabla\boldsymbol{\theta}\mathbf{n}\cdot\mathbf{n})\,\frac{\partial_{\mathrm{n}}u_0}{g} + (\nabla\boldsymbol{\theta}+\nabla\boldsymbol{\theta}^\top)\nabla u_0\cdot\mathbf{n} \in \mathrm{L}^2(\partial\Omega_0)$;
- $\partial\Omega_0 = \partial\Omega_{\mathrm{D}}^{u_0,g}\cup\partial\Omega_{\mathrm{N}}^{u_0,g}\cup\partial\Omega_{\mathrm{S-}}^{u_0,g}\cup\partial\Omega_{\mathrm{S+}}^{u_0,g}$ with

$$\begin{aligned} \partial\Omega_{\mathrm{D}}^{u_0,g} &:= \left\{s\in\partial\Omega_0,\ u_0(s)=0 \text{ and } \partial_{\mathrm{n}}u_0(s)\in(-g(s),g(s))\right\}, \\ \partial\Omega_{\mathrm{N}}^{u_0,g} &:= \left\{s\in\partial\Omega_0,\ u_0(s)\neq 0\right\}, \\ \partial\Omega_{\mathrm{S-}}^{u_0,g} &:= \left\{s\in\partial\Omega_0,\ u_0(s)=0 \text{ and } \partial_{\mathrm{n}}u_0(s)=g(s)\right\}, \\ \partial\Omega_{\mathrm{S+}}^{u_0,g} &:= \left\{s\in\partial\Omega_0,\ u_0(s)=0 \text{ and } \partial_{\mathrm{n}}u_0(s)=-g(s)\right\}. \end{aligned}$$

Main result

Theorem (Shape gradient)

The energy functional $\mathcal{J}$ admits a shape gradient at Ω_0 in the direction $\boldsymbol{\theta} \in \mathcal{C}^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ given by

$$\mathcal{J}'(\Omega_0)(\boldsymbol{\theta}) = \int_{\partial\Omega_0} \boldsymbol{\theta} \cdot \mathbf{n} \left(\frac{\|\nabla u_0\|^2 + |u_0|^2}{2} - f u_0 + Hg\,|u_0| \right. \\ \left. - \partial_{\mathrm{n}} \left(u_0 \partial_{\mathrm{n}} u_0\right) + g u_0 \nabla \left(\frac{\partial_{\mathrm{n}} u_0}{g} \right) \cdot \mathbf{n} \right),$$

where H is the mean curvature of $\partial\Omega_0$.

Proof

Let $\boldsymbol{\theta} \in \mathcal{C}^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, $t > 0$ and $\Omega_t := (\mathbf{id} + t\boldsymbol{\theta})(\Omega_0) \in \mathcal{U}_{ad}$. Then

$$\mathcal{J}((\mathbf{id} + t\boldsymbol{\theta})(\Omega_0)) = -\frac{1}{2}\int_{\Omega_0}\left(\left\|\left(\mathrm{I} + t\nabla\theta^{\top}\right)\nabla\overline{u}_t\right\|^2 + |\overline{u}_t|^2\right)\mathrm{J}_t$$

$$= \mathcal{J}(\Omega_0)\underbrace{-\frac{1}{2}\int_{\Omega_0}\left(\|\nabla u_0\|^2 + |u_0|^2\right)\mathrm{div}(\boldsymbol{\theta}) + \int_{\Omega_0}\nabla u_0 \cdot \nabla\boldsymbol{\theta}\nabla u_0 + \left\langle \overline{u}_0', u_0\right\rangle_{\mathrm{H}^1(\Omega_0)}}_{\mathcal{J}'(\Omega_0)(\theta)} + o(t).$$

Proof

$$\langle \overline{u}_0', v - \overline{u}_0' \rangle_{\mathrm{H}^1(\Omega_0)} \geq \int_{\Omega_0} -\Delta\left(\boldsymbol{\theta} \cdot \nabla u_0\right)\left(v - \overline{u}_0'\right) + \int_{\Omega_0} \boldsymbol{\theta} \cdot \nabla u_0 \left(v - \overline{u}_0'\right) + \int_{\partial\Omega_0} h^m(\boldsymbol{\theta})\left(v - \overline{u}_0'\right),$$

$$\forall v \in \mathcal{C}_{u_0, \frac{\partial_{\mathrm{n}}(F_0 - u_0)}{g}} := \left\{ w \in \mathrm{H}^1(\Omega_0) \mid w(s) \leq 0 \text{ on } \partial\Omega_{S-}^{u_0,g}, w(s) \geq 0 \text{ on } \partial\Omega_{S+}^{u_0,g}, w(s) = 0 \text{ on } \partial\Omega_{\mathrm{D}}^{u_0,g} \right\}.$$

$$\begin{aligned}
\partial\Omega_{\mathrm{D}}^{u_0,g} &= \left\{ s \in \partial\Omega_0,\ u_0(s) = 0 \text{ and } \partial_{\mathrm{n}} u_0(s) \in (-g(s), g(s)) \right\},\\
\partial\Omega_{\mathrm{S-}}^{u_0,g} &= \left\{ s \in \partial\Omega_0,\ u_0(s) = 0 \text{ and } \partial_{\mathrm{n}} u_0(s) = g(s) \right\},\\
\partial\Omega_{\mathrm{S+}}^{u_0,g} &= \left\{ s \in \partial\Omega_0,\ u_0(s) = 0 \text{ and } \partial_{\mathrm{n}} u_0(s) = -g(s) \right\}.
\end{aligned}$$

- Since $\overline{u}_0' \pm u_0 \in \mathcal{C}_{u_0, \frac{\partial_{\mathrm{n}}(F_0 - u_0)}{g}}$, then

$$\langle \overline{u}_0', u_0 \rangle_{\mathrm{H}^1(\Omega_0)} = \int_{\Omega_0} -\Delta\left(\boldsymbol{\theta} \cdot \nabla u_0\right) u_0 + \int_{\Omega_0} u_0 \boldsymbol{\theta} \cdot \nabla u_0 + \int_{\partial\Omega_0} u_0 h^m(\boldsymbol{\theta}).$$

Summary

$$\min_{\substack{\Omega \in \mathcal{U}_{ad} \\ |\Omega| = \lambda}} J(\Omega),$$

where

$$\mathcal{U}_{ad} := \Big\{ \Omega \subset \mathbb{R}^d \mid \Omega \text{ nonempty connected bounded open subset of } \mathbb{R}^d \\ \text{with } \mathcal{C}^3\text{-boundary} \Big\},$$

with the volume constraint $|\Omega| = \lambda > 0$.

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Starting from an initial shape Ω_0, then we solve numerically

$$\begin{cases} -\Delta\boldsymbol{\theta_0} + \boldsymbol{\theta_0} = 0 & \text{in } \Omega_0, \\ \nabla\boldsymbol{\theta_0}\mathbf{n} = -\left(\frac{\|\nabla u_0\|^2+|u_0|^2}{2} - fu_0 + Hg\,|u_0| - \partial_{\mathrm{n}}\left(u_0\partial_{\mathrm{n}}u_0\right) + gu_0\nabla\left(\frac{\partial_{\mathrm{n}}u_0}{g}\right)\cdot\mathbf{n} + p_0\right)\mathbf{n} & \text{on } \partial\Omega_0, \end{cases}$$

then $\Omega_1 := (\mathbf{id} + \tau\boldsymbol{\theta_0})(\Omega_0)$.

Example

- Let us consider the dimension $d = 2$.

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Example

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- The volume constraint $\lambda = \pi$.
- The initial shape Ω_0 is an ellipse with semi-major axis $a = 1.3$ and semi-minor axis $b = 1/a$.
- The source term $f \in \mathrm{H}^1(\mathbb{R}^2)$ is defined by

$$\begin{array}{rccc} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & f(x,y) := \dfrac{5 - x^2 - y^2 + xy}{4}\eta, \end{array}$$

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- The friction threshold $g_\gamma \in \mathrm{H}^2(\mathbb{R}^2)$ is defined by

$$\begin{array}{rccl} g_\gamma : & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & g_\gamma(x,y) := \gamma \left(1 + \dfrac{(\sin x)^2}{0.8}\right)\eta, \end{array}$$

with $\gamma > 0$ andt η a cut-off function.

Optimale shape (1)

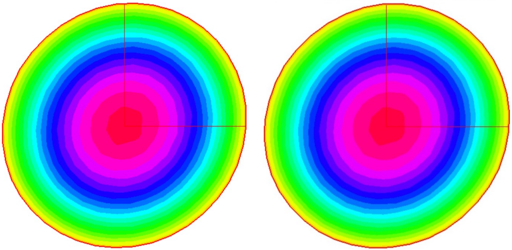

Figure: $\gamma = 0.49$. Left: Optimal shape of the Tresca friction problem. Right: Optimal shape of the homogeneous Dirichlet problem.

$$\left\{ \begin{array}{rcll} |\partial_{\mathrm{n}} u| & \leq & g_\gamma, & \\ \text{if } |\partial_{\mathrm{n}} u| & < & g_\gamma, & \text{then } u = 0; \\ \text{if } |\partial_{\mathrm{n}} u| & = & g_\gamma, & \text{then it exists } \lambda \geq 0 \text{ such as } u = -\lambda \partial_{\mathrm{n}} u. \end{array} \right.$$

Optimal shape (2)

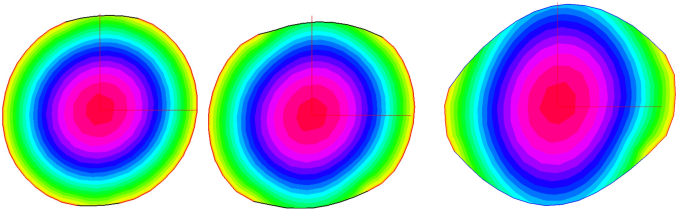

Figure: Optimal shape of the Tresca friction problem for $\gamma = 0.43, 0.37, 0.31$. The red boundary is for $u = 0$ and the black boundary for $|\partial_{\mathrm{n}} u| = g_{\gamma}$.

Optimal shape (3)

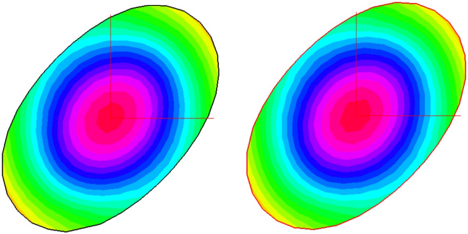

Figure: $\gamma = 0.01$. left: Optimal shape of the Tresca friction problem. right: Optimal shape of the homogeneous Neumann problem.

Thanks for your attention

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Second order epi-derivative of Φ

$$\mathrm{D}_e^2\Phi(u_0|F_0-u_0)(w) = \iota_{\mathcal{K}_{u_0,\frac{\partial_{\mathrm{n}}(F_0-u_0)}{g}}}(w)$$

$$+\int_{\Gamma_0}\left(\nabla g(s)\cdot\boldsymbol{\theta}(s)+g(s)\left(\mathrm{div}(\boldsymbol{\theta}(s))-\nabla\boldsymbol{\theta}(s)\mathbf{n}(s)\cdot\mathbf{n}(s)\right)\right)\frac{\partial_{\mathrm{n}}(F_0-u_0)(s)}{g(s)}w(s)\mathrm{d}s,$$

for all $w\in\mathrm{H}^1(\Omega_0)$, where $\mathcal{C}_{u_0,\frac{\partial_{\mathrm{n}}(F_0-u_0)}{g}}$ is the nonempty closed convex subset of $\mathrm{H}^1(\Omega_0)$ defined by

$$\mathcal{C}_{u_0,\frac{\partial_{\mathrm{n}}(F_0-u_0)}{g}} := \left\{w\in\mathrm{H}^1(\Omega_0)\mid w(s)\in\mathrm{K}_{u_0(s),\frac{\partial_{\mathrm{n}}(E_0-u_0)(s)}{g(s)}}\text{ for almost all }s\in\partial\Omega_0\right\},$$

which is also

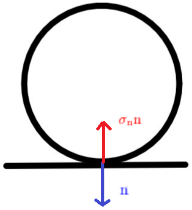
$$\mathcal{C}_{u_0,\frac{\partial_{\mathrm{n}}(F_0-u_0)}{g}} =$$
$$\left\{w\in\mathrm{H}^1(\Omega_0)\mid w(s)\leq 0\text{ on }\partial\Omega_{S-}^{u_0,g}, w(s)\geq 0\text{ on }\partial\Omega_{S+}^{u_0,g}, w(s)=0\text{ on }\partial\Omega_{\mathrm{D}}^{u_0,g}\right\}$$

Definition (Signorini's law)

Let Ω defined as above and $\partial\Omega_{\mathrm{S}} \subset \partial\Omega$. The Signorini's law on $\partial\Omega_{\mathrm{S}}$ is

$$u_{\mathrm{n}} \leq 0,\ \sigma_{\mathrm{n}} \leq 0 \text{ and } u_{\mathrm{n}}\sigma_{\mathrm{n}} = 0 \text{ on } \partial\Omega_{\mathrm{S}},$$

where σ_{n} is the normal stress.



In a scalar case :

$$u \leq 0,\ \partial_{\mathrm{n}} u \leq 0 \text{ and } u\partial_{\mathrm{n}} u = 0 \text{ on } \partial\Omega_{\mathrm{S}}.$$

The Signorini's problem:

$$\begin{cases} -\Delta u + u &= & f & \text{in } \Omega, \\ u &= & 0 & \text{on } \partial\Omega_{\mathrm{D}}, \\ \partial_{\mathrm{n}} u &= & h & \text{on } \partial\Omega_{\mathrm{N}}, \\ u \leq 0, \partial_{\mathrm{n}} u \leq h \text{ and } u\,(\partial_{\mathrm{n}} u - h) &= & 0 & \text{on } \partial\Omega_{\mathrm{S}-}, \\ u \geq 0, \partial_{\mathrm{n}} u \geq h \text{ and } u\,(\partial_{\mathrm{n}} u - h) &= & 0 & \text{on } \partial\Omega_{\mathrm{S}+}, \end{cases}$$

where $\partial\Omega_{\mathrm{D}}$, $\partial\Omega_{\mathrm{N}}$, $\partial\Omega_{\mathrm{S}-}$, $\partial\Omega_{\mathrm{S}+}$ are four measurable pairwise disjoint subsets of $\partial\Omega$ such that $\partial\Omega = \partial\Omega_{\mathrm{D}} \cup \partial\Omega_{\mathrm{N}} \cup \partial\Omega_{\mathrm{S}-} \cup \partial\Omega_{\mathrm{S}+}$, and with $f \in \mathrm{H}^1(\mathbb{R}^d)$ and $h \in \mathrm{L}^2(\partial\Omega)$.

Definition (Strong solution to the scalar Signorini problem)

A (strong) solution to the scalar Signorini problem is a function $u \in \mathrm{H}^1(\Omega)$ such that $-\Delta u + u = f$ in $\mathcal{C}_0^\infty(\Omega)'$, $u = w$ a.e. on $\partial\Omega_{\mathrm{D}}$, and also $\partial_{\mathrm{n}} u \in \mathrm{L}^2(\partial\Omega)$ with $\partial_{\mathrm{n}} u = \ell$ a.e. on $\partial\Omega_{\mathrm{N}}$, $u \leq w$, $\partial_{\mathrm{n}} u \leq \ell$ and $(u - w)(\partial_{\mathrm{n}} u - \ell) = 0$ a.e. on $\partial\Omega_{\mathrm{S}-}$, $u \geq w$, $\partial_{\mathrm{n}} u \geq \ell$ and $(u - w)(\partial_{\mathrm{n}} u - \ell) = 0$ a.e. on $\partial\Omega_{\mathrm{S}+}$.

Definition (Weak solution to the Signorini's problem)

A weak solution to the Signorini problem is a function $u \in \mathcal{K}^1_w(\Omega)$ *such that*

$$\int_\Omega \nabla u \cdot \nabla(v-u) + \int_\Omega u(v-u) \geq \int_\Omega f(v-u) + \int_{\partial\Omega} h(v-u),$$

where $\mathcal{K}^1(\Omega)$ *is the nonempty closed convex subset of* $\mathrm{H}^1(\Omega)$ *defined by*

$$\mathcal{K}^1(\Omega) := \left\{ v \in \mathrm{H}^1(\Omega),\ v \leq 0 \textit{ on } \partial\Omega_{\mathrm{S}-},\ v = 0 \textit{ on } \partial\Omega_{\mathrm{D}} \textit{ et } v \geq 0 \textit{ on } \partial\Omega_{\mathrm{S}+} \right\}.$$

Proposition (Existence and uniqueness)

The scalar Signorini problem admits a unique weak solution $u \in \mathrm{H}^1(\Omega)$ *characterized by*

$$u = \mathrm{proj}_{\mathcal{K}^1(\Omega)}(F),$$

where F *is the unique solution to the Neumann problem*

$$\begin{cases} -\Delta F + F = f & \text{in } \Omega, \\ \partial_{\mathrm{n}} F = h & \text{on } \partial\Omega, \end{cases}$$

with $\mathrm{proj}_{\mathcal{K}^1(\Omega)}$ *is the classical projection operator onto* $\mathcal{K}^1(\Omega)$ *of* $\mathrm{H}^1(\Omega)$ *for the usual scalar product* $\langle \cdot, \cdot \rangle_{\mathrm{H}^1(\Omega)}$.

Definition (Consistent decomposition)

The decomposition $\partial\Omega = \partial\Omega_{\mathrm{N}} \cup \partial\Omega_{\mathrm{D}} \cup \partial\Omega_{S-} \cup \partial\Omega_{S+}$ is said to be consistent if

1. *For almost all $s \in \partial\Omega_{\mathrm{S}-}$ (resp. $\partial\Omega_{\mathrm{S}+}$), $s \in \mathrm{int}_{\partial\Omega}(\partial\Omega_{\mathrm{S}-})$ (resp. $s \in \mathrm{int}_{\partial\Omega}(\partial\Omega_{\mathrm{S}+})$), where the notation $\mathrm{int}_{\partial\Omega}$ stands for the interior relative to $\partial\Omega$;*
2. *The nonempty closed convex subset $\mathcal{K}_w^{1/2}(\partial\Omega)$ of $\mathrm{H}^{1/2}(\partial\Omega)$ defined by*

$$\mathcal{K}_w^{1/2}(\partial\Omega) := \{\, v \in \mathrm{H}^{1/2}(\partial\Omega) \mid v \leq w \text{ a.e. on } \partial\Omega_{\mathrm{S}-}, \\ v = w \text{ a.e. on } \partial\Omega_{\mathrm{D}} \text{ and } v \geq w \text{ a.e.on } \partial\Omega_{\mathrm{S}+} \,\},$$

is dense in the nonempty closed convex subset $\mathcal{K}_w^{0}(\partial\Omega)$ of $\mathrm{L}^2(\partial\Omega)$ defined by

$$\mathcal{K}_w^{0}(\partial\Omega) := \{\, v \in \mathrm{L}^2(\partial\Omega) \mid v \leq w \text{ a.e. on } \partial\Omega_{\mathrm{S}-}, \\ v = w \text{ a.e. on } \partial\Omega_{\mathrm{D}} \text{ and } v \geq w \text{ a.e. on } \partial\Omega_{\mathrm{S}+} \,\}.$$

Definition (Strong solution to the Tresca friction problem)

A strong solution to the Tresca friction problem is a function $u \in \mathrm{H}^1(\Omega)$*, such that* $-\Delta u + u = f$ *in* $\mathcal{D}'(\Omega)$*,* $\partial_{\mathrm{n}} u \in \mathrm{L}^2(\partial\Omega)$*,* $|\partial_{\mathrm{n}} u(s)| \leq g(s)$ *and* $u(s)\partial_{\mathrm{n}} u(s) = -g(s)|u(s)|$ *for almost all* $s \in \partial\Omega$*.*

Definition (Mosco convergence)

The outer, weak-outer, inner and weak-inner limits of a parameterized family $(S_t)_{t>0}$ of subsets of $\mathcal{H}$ are respectively defined by

$$\begin{aligned}
\limsup S_t &:= \left\{x \in \mathcal{H} \mid \exists (t_n)_{n\in\mathbb{N}} \to 0^+,\ \exists\,(x_n)_{n\in\mathbb{N}} \to x,\ \forall n \in \mathbb{N},\ x_n \in S_{t_n}\right\},\\
\text{w-}\limsup S_t &:= \left\{x \in \mathcal{H} \mid \exists (t_n)_{n\in\mathbb{N}} \to 0^+,\ \exists\,(x_n)_{n\in\mathbb{N}} \rightharpoonup x,\ \forall n \in \mathbb{N},\ x_n \in S_{t_n}\right\},\\
\liminf S_t &:= \left\{x \in \mathcal{H} \mid \forall (t_n)_{n\in\mathbb{N}} \to 0^+,\ \exists\,(x_n)_{n\in\mathbb{N}} \to x,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ x_n \in S_{t_n}\right\},\\
\text{w-}\liminf S_t &:= \left\{x \in \mathcal{H} \mid \forall (t_n)_{n\in\mathbb{N}} \to 0^+,\ \exists\,(x_n)_{n\in\mathbb{N}} \rightharpoonup x,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ x_n \in S_{t_n}\right\}.
\end{aligned}$$

The family $(S_t)_{t>0}$ is said to be Mosco convergent if

$$\text{w-}\limsup S_t \subset \liminf S_t.$$

In that case all the previous limits are equal and we write

$$\text{M-}\lim\ S_t := \liminf S_t = \limsup S_t = \text{w-}\liminf S_t = \text{w-}\limsup S_t.$$

Definition (Mosco epi-convergence)

Let $(f_t)_{t>0}$ *be a parameterized family of functions* $f_t : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$ *for all* $t > 0$*. We say that* $(f_t)_{t>0}$ *is Mosco epi-convergent if* $(\mathrm{epi}(f_t))_{t>0}$ *is Mosco convergent in* $\mathcal{H} \times \mathbb{R}$*. Then we denote by* $\mathrm{ME\text{-}lim}\ f_t : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$ *the function characterized by its epigraph* $\mathrm{epi}\,(\mathrm{ME\text{-}lim}\ f_t) := \mathrm{M\text{-}lim}\ \mathrm{epi}\,(f_t)$ *and we say that* $(f_t)_{t>0}$ *Mosco epi-converges to* $\mathrm{ME\text{-}lim}\ f_t$*.*

For all $u \in \mathrm{H}^1(\Omega)$, let us consider the problem

$$\begin{cases} -\Delta v + v = 0 & \text{in } \Omega, \\ \partial_{\mathrm{n}} v(s) \in \partial g(s)|\cdot|(u(s)) & \text{on } \partial\Omega, \end{cases} \tag{2}$$

A solution is a function function $v \in \mathrm{H}^1(\Omega)$ such that $-\Delta v + v = 0$ in $\mathcal{D}'(\Omega)$, $\partial_{\mathrm{n}} v \in \mathrm{L}^2(\partial\Omega)$ and $\partial_{\mathrm{n}} v(s) \in \partial g(s)|\cdot|(u(s))$ for almost all $s \in \partial\Omega$.

Lemma

Let $u \in \mathrm{H}^1(\Omega)$. Then,

$$\partial\Phi(0,\cdot)(u) = \textit{the set of solutions to Problem (2)},$$

where $\Phi(0,\cdot)$ is the parameterized Tresca friction funtional at $t = 0$.

Proposition

Let us consider $\boldsymbol{\theta} \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$ *and* $v \in \mathrm{W}^{2,1}(\Omega)$*. Then*

$$\int_{\partial\Omega} (\boldsymbol{\theta} \cdot \nabla v + v \mathrm{div}(\boldsymbol{\theta}) - v(\nabla \boldsymbol{\theta} \mathbf{n} \cdot \mathbf{n})) = \int_{\partial\Omega} \boldsymbol{\theta} \cdot \mathbf{n}(\partial_{\mathrm{n}} v + Hv).$$

Proposition

Let $v \in \mathrm{H}^1(\Omega)$ *be such that* $\Delta v \in \mathrm{L}^2(\Omega)$*, and consider* $\boldsymbol{V} \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d) \cap \mathrm{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$*. Then*

$$\Delta (\boldsymbol{V} \cdot \nabla v) = \mathrm{div}\left((\Delta v)\, \boldsymbol{V} - \mathrm{div}(\boldsymbol{V}) \nabla v + (\nabla \boldsymbol{V} + \nabla \boldsymbol{V}^\top) \nabla v \right) \qquad \text{in } \mathcal{C}_0^\infty(\Omega)'.$$

Theorem (Shape derivative)

The shape derivative defined by $u_0' := \overline{u}_0' - \nabla u_0 \cdot \boldsymbol{\theta} \in \mathrm{H}^1(\Omega_0)$, *is the unique weak solution to the Signorini's problem*

$$\begin{cases} -\Delta u_0' + u_0' & = & 0 & \text{in } \Omega_0, \\ u_0' & = & -\boldsymbol{\theta} \cdot \nabla u_0 & \text{on } \partial\Omega_{\mathrm{D}}^{u_0,g}, \\ \partial_{\mathrm{n}} u_0' & = & h^s(\boldsymbol{\theta}) & \text{on } \partial\Omega_{\mathrm{N}}^{u_0,g}, \\ u_0' \leq -\boldsymbol{\theta} \cdot \nabla u_0,\ \partial_{\mathrm{n}} u_0' \leq h^s(\boldsymbol{\theta}) \text{ and } \left(u_0' + \boldsymbol{\theta} \cdot \nabla u_0\right)\left(\partial_{\mathrm{n}} u_0' - h^s(\boldsymbol{\theta})\right) & = & 0 & \text{on } \partial\Omega_{\mathrm{S-}}^{u_0,g}, \\ u_0' \geq -\boldsymbol{\theta} \cdot \nabla u_0,\ \partial_{\mathrm{n}} u_0' \geq h^s(\boldsymbol{\theta}) \text{ and } \left(u_0' + \boldsymbol{\theta} \cdot \nabla u_0\right)\left(\partial_{\mathrm{n}} u_0' - h^s(\boldsymbol{\theta})\right) & = & 0 & \text{on } \partial\Omega_{\mathrm{S+}}^{u_0,g}, \end{cases}$$

où $h^s(\boldsymbol{\theta}) := \nabla_\Gamma u_0 \cdot \nabla_\Gamma(\boldsymbol{\theta} \cdot \mathbf{n}) - g\nabla\left(\frac{\partial_{\mathrm{n}} u_0}{g}\right) \cdot \boldsymbol{\theta} \in \mathrm{L}^2(\Gamma_0)$

least-square functional

Let us consider the least-square functional

$$J(\Omega) = \int_{\Omega} |u(\Omega) - w|^2 \,,$$

with $w \in \mathrm{H}^1(\mathbb{R}^d)$ a target displacement. Thus one gets

$$J'(\Omega_0)(\boldsymbol{\theta}) = \int_{\partial\Omega_0} |u_0 - w|^2 \, \boldsymbol{\theta}\cdot\mathrm{n} - \int_{\Omega_0} 2(u_0 - w)\nabla u_0 \cdot \boldsymbol{\theta} + \int_{\Omega_0} 2(u_0 - w)\overline{u}_0'.$$

with $\overline{u}_0'$ the material derivative.

least-square functional

$$\begin{cases} -\Delta p + p = -2(u_0 - w) & \text{in } \Omega_0, \\ p = 0 & \text{on } \partial\Omega_{\mathrm{D}}^{u_0,g}, \\ \partial_{\mathrm{n}} p = 0 & \text{on } \partial\Omega_{\mathrm{N}}^{u_0,g}, \\ p \leq 0,\ \partial_{\mathrm{n}} p \geq 0 \text{ and } p\partial_{\mathrm{n}} p = 0 & \text{on } \partial\Omega_{\mathrm{S}-}^{u_0,g}, \\ p \geq 0,\ \partial_{\mathrm{n}} p \leq 0 \text{ and } p\partial_{\mathrm{n}} p = 0 & \text{on } \partial\Omega_{\mathrm{S}+}^{u_0,g}, \end{cases}$$

Weak formulation:

$$\langle p, \varphi - p \rangle_{\mathrm{H}^1(\Omega_0)} \leq \int_{\Omega_0} -2(u_0 - w)(p - \varphi), \qquad \forall \varphi \in \mathcal{C}_{u_0, \frac{\partial_{\mathrm{n}}(E_0 - u_0)}{g}}.$$

least-square functional

$$J'(\Omega_0)(\boldsymbol{\theta}) \leq \int_{\partial\Omega_0} \boldsymbol{\theta}\cdot\mathbf{n}\left(|u_0 - w|^2 + p(u_0 - f) + \nabla p\cdot\nabla u_0 - Hp\partial_{\mathrm{n}} u_0 - \partial_{\mathrm{n}}(p\partial_{\mathrm{n}} u_0) + pg\nabla\left(\frac{\partial u_0}{g}\right)\cdot\mathbf{n}\right).$$

Particular case of the second-order epi-derivative

Let us denote

$$\partial\Omega_{\mathrm{N}}^{u_0,g} := \{s \in \partial\Omega_0,\ u_0(s) \neq 0\}.$$

- $\partial\Omega_{\mathrm{N}}^{u_0,g}$ has a null measure;
- If $|u_0| \geq C$ a.e. on $\partial\Omega_{\mathrm{N}}^{u_0,g}$, with $C > 0$;
- If the dimension $d = 2$, $\partial\Omega$ is diffeomorphic to the unit disk, u_0 and $\partial_{\mathrm{n}} u_0$ are continuous on $\partial\Omega$.