On the computation of the B-differential of the Min C-function for the balanced linear complementarity problem

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Outline

Plan

Form of the problems

Complementarity problems [CPS92; FP03]

$$\begin{aligned} & 0 \leq F(x) \perp G(x) \geq 0 \\ \Leftrightarrow \forall\, i,\ & F_i(x) \geq 0, G_i(x) \geq 0, F_i(x)G_i(x) = 0 \end{aligned} \tag{1}$$

Where $F, G : \mathbb{R}^n \to \mathbb{R}^n$ are smooth. Affine case:

$$\begin{aligned} F(x) \equiv Ax + a,\ \ G(x) \equiv Bx + b, \quad A, B \in \mathbb{R}^{n \times n}, a, b \in \mathbb{R}^n \\ 0 \leq (Ax + a) \perp (Bx + b) \geq 0 \end{aligned} \tag{2}$$

Remark: $u \geq 0, v \geq 0, uv = 0 \Leftrightarrow \min(u, v) = 0$

(1) $\Leftrightarrow \forall\, i, H_i(x) := \min(F_i(x), G_i(x)) = 0 \Leftrightarrow H(x) = 0$

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Reformulation by C-functions

C-functions

$$\varphi(u,v) = 0 \Leftrightarrow u \geq 0, v \geq 0, uv = 0$$

Examples: minimum, Fischer,... applied componentwise.

Fischer $\varphi_F(u,v) = \sqrt{u^2+v^2} - (u+v)$ [Fis92]; more differentiable, less linear.
φ_F: much work already done [FS97; GK96]

C-functions are nondifferentiable $\rightarrow$ nonsmooth techniques

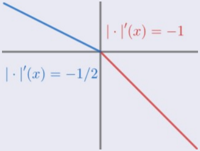
Nonsmooth equations - context

For scalar functions: subgradients
For systems $H(x) = 0$: semismooth Newton [QS93]

Generalized derivatives: Bouligand differential

$$\partial_B H(x) = \{J \in \mathbb{R}^{n\times n} : \exists\ (x_k)_k \to x, H'(x_k) \text{ exists and} \to J\} \quad (3)$$

Example: $h(x) = \min(-x/2, -x)$,
$\partial_B h(0) = \{-1/2, -1\}$.
One element $J^0 \in \partial_B H$: technique of [Qi93]



also: Clarke $\partial_C H(x) = \mathrm{conv}(\partial_B H(x))$ $(\to [-1, -1/2])$

Main difficulty

First part of this work

Determine generalized Jacobians of $x \mapsto \min(Ax + a, Bx + b)$

- structure of the Jacobians?
- $|\partial_B H(x)|$? finite but exponential
- how to get them efficiently?

Difficulties already seen previously, for instance [CX11].

Plan

B-differential: which sequences to use?

$$\partial_B H(x) = \{J : \exists\ (x_k)_k \to x, H'(x_k) \text{ defined and} \to J\} \quad (x_k)_k =?$$

H piecewise affine, H' piecewise constant

- $A_{i:} = B_{i:}$, H_i always the same ($A_{i:}$ or $B_{i:}$ term)
- $A_{i:}x + a_i < B_{i:}x + b_i \Rightarrow$ true for x_k close to x $H_i'(x) = A_{i:}$
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If any holds $\Rightarrow H_i$ differentiable around x.
J_i known and the same for all J's

$$I(x) := \{i \in [1 : n] : A_{i:}x + a_i = B_{i:}x + b_i, A_{i:} \neq B_{i:}\}$$

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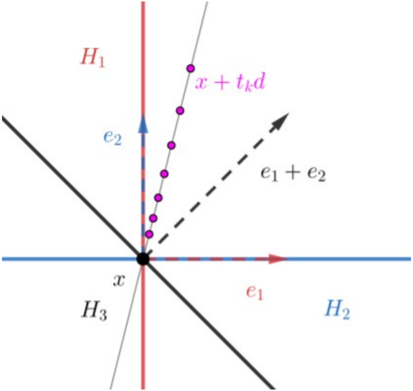
Vectors and hyperplanes - 1

Affine functions are equal on hyperplanes that pass through x.



In red, blue and black are some hyperplanes and their normal vectors. In magenta, points of the form $x + t_k d$, $t_k \searrow 0$. They stay in a "region", so J is constant: sequences can be reduced to points.

Vectors and hyperplanes - 2

$|I(x)| = m$ hyperplanes, $H_i = v_i^\perp, v_i^\mathsf{T} = B_{i:} - A_{i:}$
rest = connected sets (regions) on $+$ or $-$ side of every H_i.

Fundamental question

$$\text{given } v_i := (B_{i:} - A_{i:})^\mathsf{T}$$
$$\text{find all } s = (s_1, \ldots, s_m) \in \{\pm 1\}^m,$$
$$\text{s.t. } \exists\ d_s, \forall\ i \in [1:m], s_i v_i^\mathsf{T} d_s > 0$$

2^m linear feasibility pbs to solve... How to improve?

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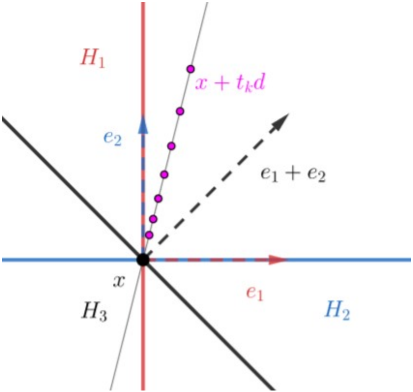

Magenta points : $+$ side of every hyperplane. Their J is $J_i = A_{i:}$ $[+ \Leftrightarrow A_{i:}, - \Leftrightarrow B_{i:}]$. However, $J_1 = A_{1:}, J_2 = A_{2:}, J_3 = B_{3:}$ is impossible: it means being over the blue, right to the red but southwest to the black.

From literature - 1

Algebraic approach: arrangement of hyperplanes

No easy answer: combinatorial problem, well-studied.
Deeper considerations, objects of smaller dimension.
Difficult : Möbius function, oriented matroids, lattice theory...
Various examples: [Zas75; AW81; CS95; Sta07]

Our case: "central", $x \in \bigcap H_i$. Formula [AM17]:

$$|\partial_B H(x)| = \sum_{T \subset \{H_i, i \in [1:m]\}} (-1)^{|T| - n + \dim(\bigcap H_t, t \in T)}$$
$$= \sum_{\mathcal{V} \subset \{v_1, \dots, v_m\}} (-1)^{|\mathcal{V}| - \operatorname{rank}(\mathcal{V})}$$

no very clear interpretation, and which Jacobian matrices?

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\end{aligned}
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From literature - 2

There exists a strong and different hypothesis $\rightarrow$ different results.
Arbitrary case also exists.

Multiple algorithms exist, operating on various angles / situations.

- dimensional recursive sweeping by hyperplanes [BN82]
- ordering of the regions into search [Sle98]
- [RČ18]...

Plan

Framework

Given m vectors, $V = [v_1, \dots, v_m] \in \mathbb{R}^{n \times m}$
Up to 2^m elements to compute, likely less.
Element $\equiv$ signs of a feasible system: how to determine the s's?
Some easy cases:

- n or $m \in \{1, 2\}$ (in fact, $\mathrm{rank}(V) = 1, 2$)
- $\mathrm{rank}(V) = 2$: $2m$ elements
- $\mathrm{rank}(V) = m$ (2^m sign vectors)

Full rank case: observed in [CX11]

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Method - adding vectors one at a time

If all sign vectors are found for $(v_1, \ldots, v_{k-1})$: for $(v_1, \ldots, v_k)$?

With one more vector

- Given $(v_1, \ldots, v_{k-1})$; v_k ; $\mathcal{S}_{k-1} \subseteq \{\pm 1\}^{k-1}$

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- $v_k^\mathsf{T} d_s^{k-1} < 0 \Rightarrow \begin{cases} -v_k^\mathsf{T} d_s^{k-1} > 0 \\ s_i v_i^\mathsf{T} d_s^{k-1} > 0 \end{cases} \checkmark, \begin{cases} +v_k^\mathsf{T} d > 0 \\ s_i v_i^\mathsf{T} d > 0 \end{cases} ? \to \text{opt}$

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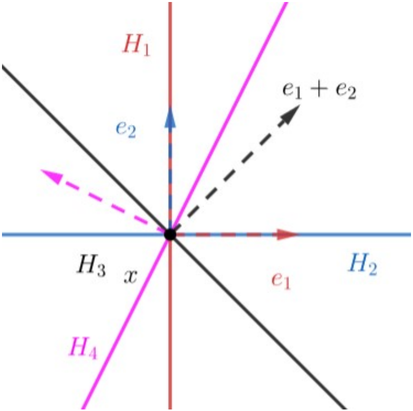
- $v_k^\mathsf{T} d_s^{k-1} = 0 \Rightarrow$ both systems $\checkmark$ by perturbation

Illustration of the method



Magenta: added hyperplane. Some of the regions, top right and bottom left ones, are split by H_4. Equivalently, the $(k-1)$-systems corresponding to these regions have their two offshoot systems feasible.

Plan

Study of the iterative subproblem - 1

- vectors $v_1, \ldots, v_{k-1}$ and associated signs $s = (s_1, \ldots, s_{k-1})$
- d_s^{k-1} s.t. $\forall\ i \in [1 : k-1], s_i v_i^\mathsf{T} d_s > 0$
- $\mathrm{sign}(v_k^\mathsf{T} d_s^{k-1}) v_k^\mathsf{T} d_s^{k-1}$ already > 0
- feasible with $s_k = -\mathrm{sign}(v_k^\mathsf{T} d_s)$? [new d?]

Linear optimization formulation

$$\exists\ d, \text{s.t.} \begin{cases} s_k v_k^\mathsf{T} d > 0 \\ s_i v_i^\mathsf{T} d > 0, \quad i \in [1 : k-1] \end{cases}$$
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Study of the iterative subproblem - 2

Hypothesis for step $k-1$: $s_i v_i^\top d > 0 \rightarrow$ known

$$\begin{cases} \inf \ -s_k v_k^\top d \\ s_i v_i^\top d \geq 0 \ \forall \ i \\ ||d|| \leq D \end{cases}$$

$$\varphi_\mu(d) = -s_k v_k^\top d - \mu \sum \log(s_i v_i^\top d) - \mu \log((D^2 - ||d||^2)/2)$$

- interior point technique
- bound constraint
- finite solution
- objective sign only
- IP proof in progress (NL constraint)

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Study of the iterative subproblem - 3

Bound constraint $\Rightarrow$ finite solution $\Rightarrow$ dual problem
Dual = projection $\leftrightarrow$ quadratic problem (QP)

Interior points

bounded domain: constraint $||d|| \leq D$ [homogeneity].
$\rightarrow$ framework of [Nes22]: complexity result
(polynomial, constants depend on solution)

Theoretical bound

The metric: number of simple subproblems ("hard things").

- Upper bound of efficiency?
- Default: next iteration can double, or "much less"
- Size / complexity of subproblems also increase.

$$\mathcal{B} = \min(2^m - 2^r, (m-r)|\mathcal{S}|), \quad |\mathcal{S}| = |\partial_B| \tag{5}$$

Also found in literature [Sle98]

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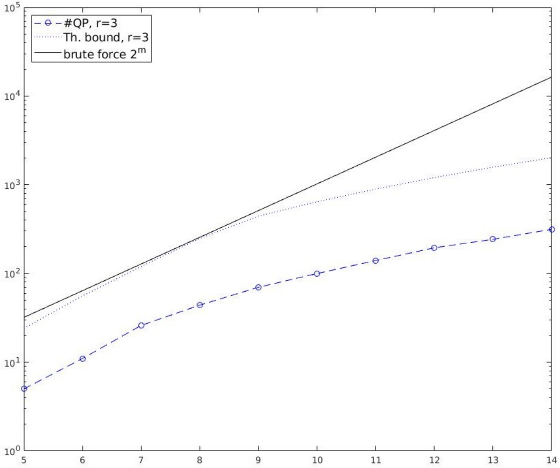
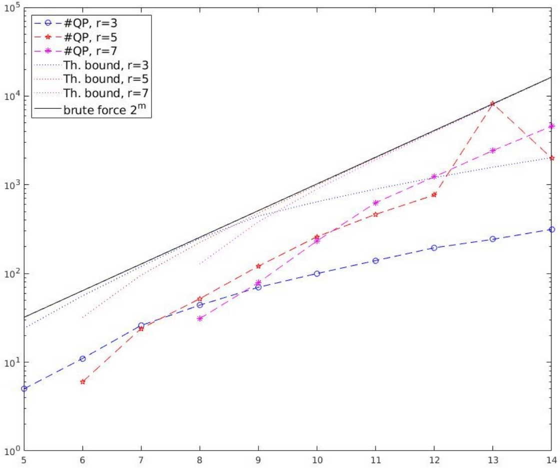


Number of problems solved for $n = 7$, $m \in [5 : 14]$, $r = 3$

Distance to the bound - 2



Number of problems solved for $n = 7$, $m \in [5:14]$, $r = 3, 5, 7$

Plan

Other illustration

Equivalent problems/representations

- Arrangement of hyperplanes with nonempty intersection
- Given some vectors, does reverting them yield a pointed cone
- How many pairs of convex subsets a set of points generate?
- Systems of inequations [chosen]
- Orthants having nonempty intersection with a null space

Linear algebra / convex analysis / combinatorics...

B-differential of the merit function - 1

Setting and result

$$\theta(x) := \frac{1}{2}||H(x)||^2$$

Differentiable case: $\nabla\theta = \nabla H \times H$; result:

$$\partial_B \theta(x) = \partial_B H(x)^\mathsf{T} \times H(x)$$

[$\subseteq$] $J \in \partial_B H(x) \to \exists\ x_k, H'(x_k) \to J$; so $\nabla H(x_k) H(x_k) \to J^\mathsf{T} H$.
[$\supseteq$] $v \in \partial_B \theta(x)$; if H is also differentiable on x_k, same thing.
Otherwise, no "$\nabla H(x_k)$".

B-differential of the merit function - 2

- subsequence: $[1:n] = I \cup I^c$: $H_i, i \in I$ non-differentiable
- $\theta = \sum_{I^c} + \sum_I$, so $\sum_I$ differentiable
- Taylor expansion of $\sum_I$

$$\Rightarrow d \mapsto \sum_I H_i^k \min(A_{i:}d, B_{i:}d) \quad \text{linear in } d$$

In fact (simple but technical proof), means that terms cancel out.
And we also have $\nabla H_I \times H_I$, so we get $\partial_B H(x)^\mathsf{T} \times H(x) = \partial_B \theta(x)$

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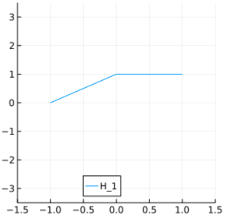
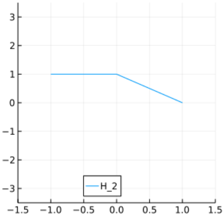
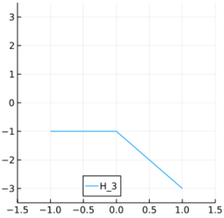
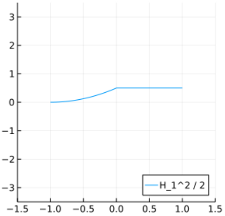
B-differential of the merit function - 3







$H_1 = \min(0, t) + 1$ $H_2 = \min(0, -t) + 1$ $H_3 = \min(0, -2t) - 1$



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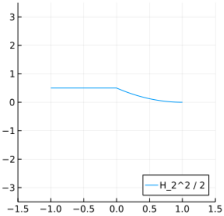

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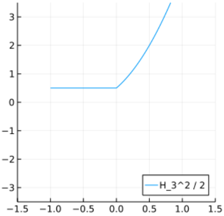
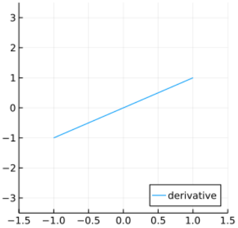
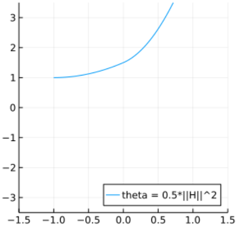


B-differential of the merit function - 4



up: the sum of coefficients $\times$ minima is linear; down: θ is differentiable



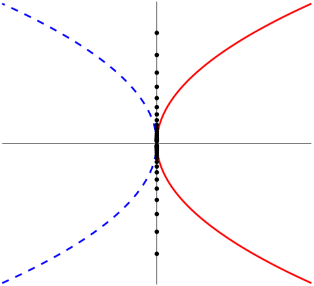
A word about the nonlinear setting

$\mathcal{L}_x H(z) = \min(F(x) + F'(x)(z - x), G(x) + G'(x)(z - x))$
In general, we have $\partial_B(\mathcal{L}_x(H))(x) \subsetneq \partial_B H(x)$.

Blue, red: nonlinear sets $F_i = G_i$.

On the vertical axis (or nearby), "new" and "nonlinear" jacobian matrices appear (x_k =dots)

NSC for equality, but technical ("stitching" the manifolds).



The manifolds are smooth but not necessarily hyperplanes anymore

Conclusion

Results

- ∂_B computed efficiently
- Less subproblems than the theoretical upper bound
- Useable for $\partial_B\theta(x)$

Open questions

- ε-issues ("$= 0$?")
- larger m, r's

Thank you for your attention! Any question?

Bibliographic elements I

[AM17] M. Aguiar and S. Mahajan. *Topics in hyperplane arrangements*. Vol. 226. Mathematical Surveyx and Monographs. AMS, 2017.

[AW81] Gerald Alexanderson and John Wetzel. "Arrangements of planes in space". In: *Discrete Mathematics* 34 (Dec. 1981). [doi], pp. 219–240.

[BN82] Hanspeter Bieri and W. Nef. "A recursive sweep-plane algorithm, determining all cells of a finite division of Rd". In: *Computing* 28 (Sept. 1982). [doi], pp. 189–198.

[CPS92] R.W. Cottle, J.-S. Pang, and R.E. Stone. *The Linear Complementarity Problem*. Academic Press, Boston, 1992.

Bibliographic elements II

[CS95] Kenneth Clarkson and Peter Shor. "Applications of Random Sampling in Computational Geometry, II". In: *Discrete and Computational Geometry* 4 (Mar. 1995). [doi], pp. 387–421.

[CX11] X. Chen and S. Xiang. "Computation of generalized differentials in nonlinear complementarity problems". In: *Computational Optimization and Applications* 50 (2011). [doi], pp. 403–423.

[Fis92] A. Fischer. "A special Newton-type optimization method". In: *Optimization* 24 (1992). [doi], pp. 269–284.

Bibliographic elements III

[FP03] F. Facchinei and J.-S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer Series in Operations Research. Springer, 2003.

[FS97] F. Facchinei and J. Soares. “A new merit function for nonlinear complementarity problems and a related algorithm”. In: *SIAM Journal on Optimization* 7.1 (1997). [doi], pp. 225–247.

[GK96] C. Geiger and C. Kanzow. “On the resolution of monotone complementarity problems”. In: *Computational Optimization and Applications* 5 (1996), pp. 155–173.

Bibliographic elements IV

[Nes22] Yurii Nesterov. "Set-Limited Functions and Polynomial-Time Interior-Point Methods". In: *LIDAM discussion Paper CORE* (Mar. 2022).

[Qi93] L. Qi. "Convergence Analysis of Some Algorithms for Solving Nonsmooth Equations". In: *Mathematics of Operations Research* 18 (Feb. 1993). [doi], pp. 227–244.

[QS93] L. Qi and J. Sun. "A nonsmooth version of Newton's method". In: *Mathematical Programming* 58 (1993). [doi], pp. 353–367.

Bibliographic elements V

[RČ18] Miroslav Rada and Michal Černý. "A New Algorithm for Enumeration of Cells of Hyperplane Arrangements and a Comparison with Avis and Fukuda's Reverse Search". In: *SIAM Journal on Discrete Mathematics* 32 (Jan. 2018), pp. 455–473. DOI: 10.1137/15M1027930.

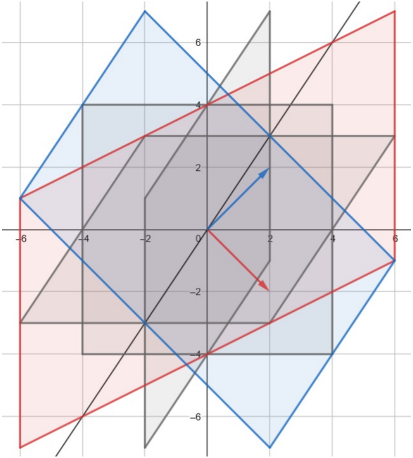
[Sle98] Nora Sleumer. "Output-Sensitive Cell Enumeration in Hyperplane Arrangements.". In: vol. 6. Jan. 1998, pp. 300–309.

[Sta07] Richard Stanley. "An introduction to hyperplane arrangements". In: Oct. 2007, pp. 389–496. ISBN: 9780821837368. DOI: 10.1090/pcms/013/08.

Bibliographic elements VI

[Zas75] T. Zaslavsky. "Facing up to Arrangements: Face-Count Formulas for Partitions of Space by Hyperplanes". In: *Memoirs of the American Mathematical Society* 154 (Jan. 1975). [doi].

Example with hyperplanes



Grey: hyperplanes orthogonal to e_1, e_2, e_3, red: $(e_1 + e_2)^\perp$, blue: $(e_2 + e_3)^\perp$

Sign vectors and directions

s	1	2	3	4	5	6	7	8	9
s_1	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
s_2	$+1$	$+1$	$+1$	-1	-1	-1	-1	-1	-1
s_3	$+1$	-1	-1	$+1$	$+1$	$+1$	$+1$	-1	-1
s_4	$+1$	$+1$	$+1$	$+1$	$+1$	-1	-1	$+1$	-1
s_5	$+1$	$+1$	-1	$+1$	-1	$+1$	-1	-1	-1
d_1	$+1$	$+1$	$+1$	$+1$	$+1$	$+1/2$	$+1/2$	$+1$	$+1/2$
d_2	$+1$	$+1$	$+1$	$-1/2$	$-3/4$	$-3/4$	-1	$-1/2$	-1
d_3	$+1$	$-1/2$	-2	$+1$	$+1/2$	$+1$	$+1/2$	$-1/2$	$-1/2$

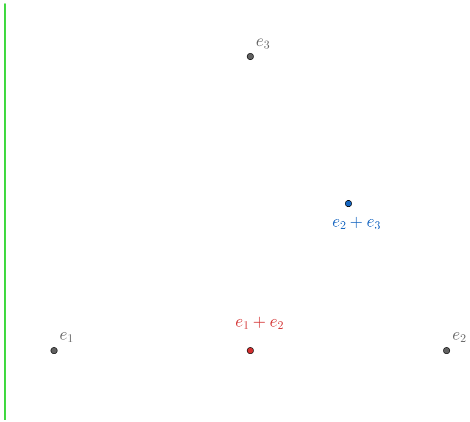
Table: Sign vectors and associated d's for the configuration $v_1 = e_1, v_2 = e_2, v_3 = e_3, v_4 = e_1 + e_2, v_5 = e_2 + e_3$ - first half

Sign vectors and directions - symmetric

s	10	11	12	13	14	15	16	17	18
s_1	-1	-1	-1	-1	-1	-1	-1	-1	-1
s_2	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	-1	-1	-1
s_3	$+1$	$+1$	-1	-1	-1	-1	$+1$	$+1$	-1
s_4	$+1$	-1	$+1$	$+1$	-1	-1	-1	-1	-1
s_5	$+1$	$+1$	$+1$	-1	$+1$	-1	$+1$	-1	-1
d_1	$-1/2$	-1	$-1/2$	$-1/2$	-1	-1	-1	-1	-1
d_2	$+1$	$+1/2$	$+1$	$+3/4$	$+3/4$	$+1/2$	-1	-1	-1
d_3	$+1/2$	$+1/2$	$-1/2$	-1	$-1/2$	-1	$+2$	$+1/2$	-1

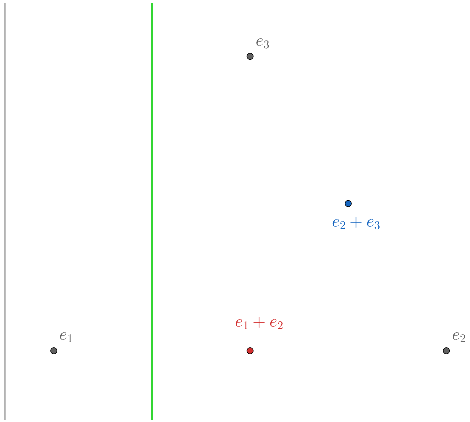
Table: Sign vectors and associated d's for the same configuration as previous table - other half, symmetric

Illustration of the convex pairs



Previous problem, the convex separation form, $s = \{+1, +1, +1, +1, +1\}$ (and symmetric $s = \{-1, -1, -1, -1, -1\}$)

Illustration of the convex pairs



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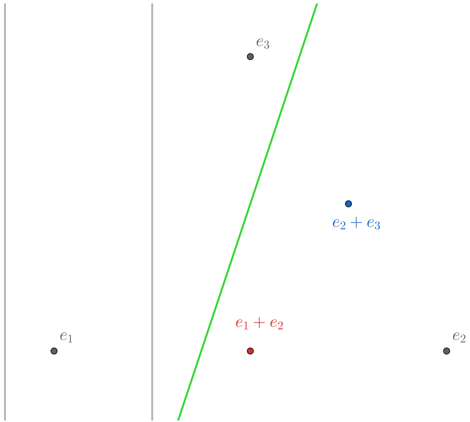

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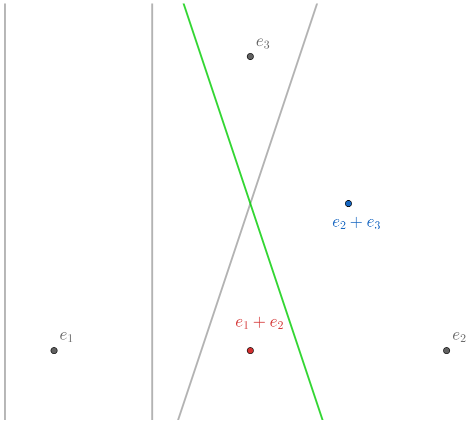

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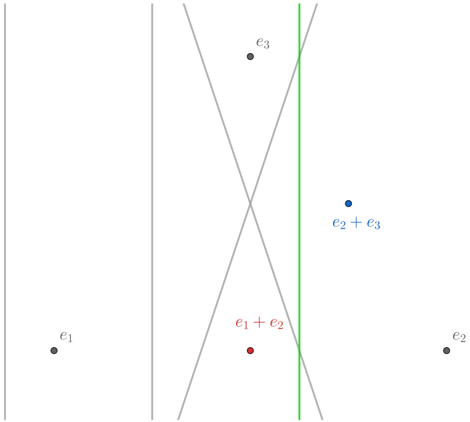

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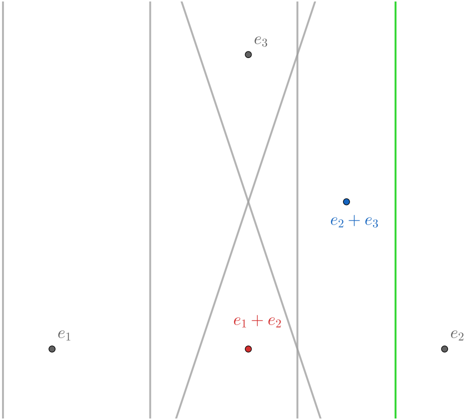

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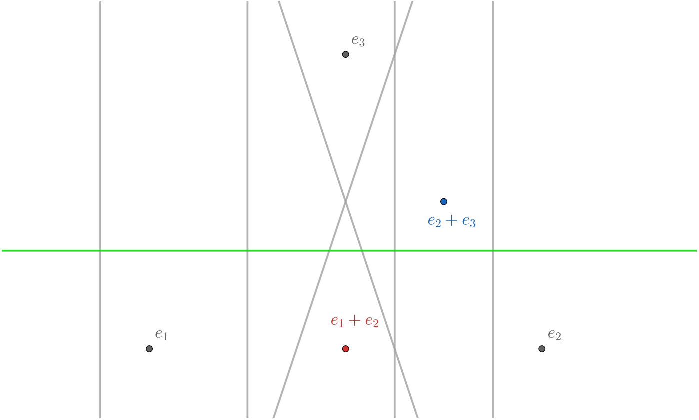

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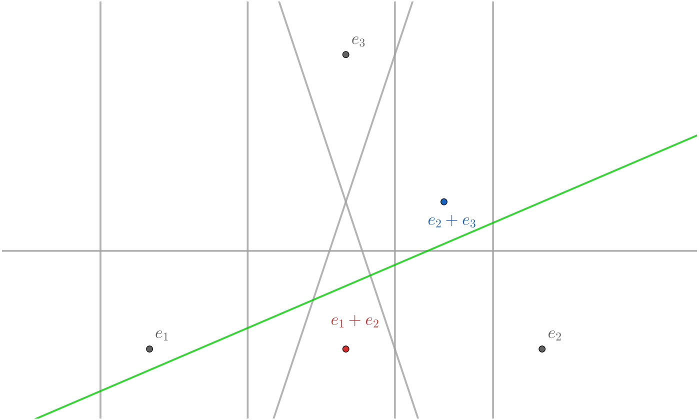

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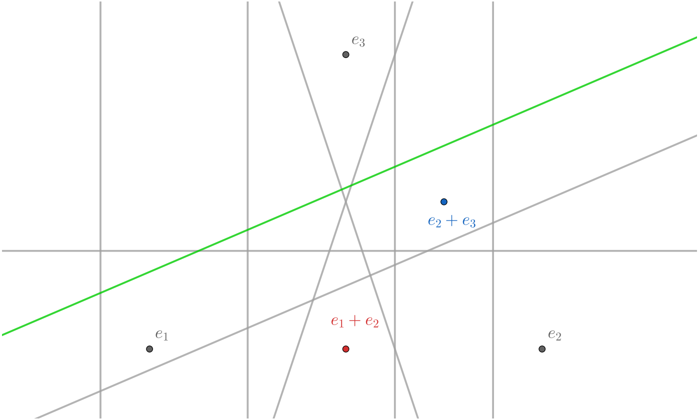

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Summary of the equivalent representations

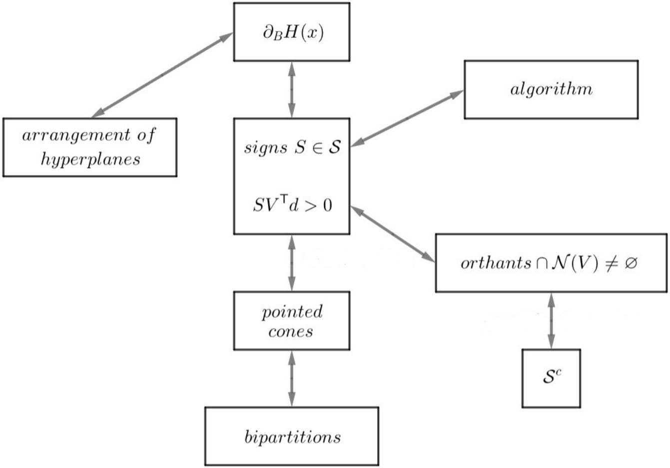

Diagram of the various reformulations

A word about the nonlinear setting - 2

A condition to get equality

$\mathcal{V}_i = \{x : F_i(x) = G_i(x)\}$ [hyperplanes $\to$ smooth manifolds]

$$\left\{ \begin{array}{l} \forall \; v_{i_0} = \sum \alpha_i v_i, \exists \, \mathcal{U} \ni x, \\ \mathcal{U} \cap \bigcap \mathcal{V}_i \subseteq \mathcal{U} \bigcap \mathcal{V}_{i_0} \end{array} \right. \Leftrightarrow \partial_B(\mathcal{L}_x H)(x) = \partial_B H(x)$$

where $\mathcal{U}$ neighborhood, $\alpha_i \neq 0, v_i$ linearly independent.

"Stitching" the manifolds (around x) to avoid "nonlinear holes". If

some equations are verified, another also is.
But needs to be done for every linear combination...